THE ENERGY-MOMENTUM TENSOR FOR THE GRAVITATIONAL FIELD.

S.V. Babak[1)] and L.P. Grishchuk [1)2)]

1) Department of Physics and Astronomy, Cardiff University, Cardiff CF24 3YB, UK.

2) Sternberg Astronomical Institute, Moscow University, Moscow 119899, Russia

E-mail: babak@astro.cf.ac.uk ; grishchuk@astro.cf.ac.uk

Abstract

The search for the gravitational energy-momentum tensor is often qualified as an attempt of looking for "the right answer to the wrong question". This position does not seem convincing to us. We think that we have found the right answer to the properly formulated question. We have further developed the field theoretical formulation of the general relativity which treats gravity as a non-linear tensor field in flat space-time. The Minkowski metric is a reflection of experimental facts, not a possible choice of the artificial "prior geometry". In this approach, we have arrived at the gravitational energy-momentum tensor which is: 1) derivable from the Lagrangian in a regular prescribed way, 2) tensor under arbitrary coordinate transformations, 3) symmetric in its components, 4) conserved due to the equations of motion derived from the same Lagrangian, 5) free of the second (highest) derivatives of the field variables, and 6) is unique up to trivial modifications not containing the field variables. There is nothing else, in addition to these 6 conditions, that one could demand from an energy-momentum object, acceptable both on physical and mathematical grounds. The derived gravitational energy-momentum tensor should be useful in practical applications.

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I. INTRODUCTION

The notions of energy and momentum play important role in physics [1], [2]. These quantities are useful because they are conserved. The conservation laws follow from the equations of motion, but we can gain important information about the system even without explicitely solving its equations of motion.

For a distributed system (or a field) the densities of energy, momentum, and flux of momentum are functions of points labelled by some coordinates x^α. These functions combine in the energy-momentum tensor $T^{\mu\nu}(x^\alpha)$, that is, the components of $T^{\mu\nu}$ transform according to the tensor rule under arbitrary transformations of the coordinates x^α (independently of whether the space of points x^α is endowed with one or another metric tensor). It would be embarrassing to use an energy-momentum object which did not transform as a tensor under, say, a transition from rectangular to spherical coordinates. Usually, the $T^{\mu\nu}$ is a symmetric tensor, $T^{\mu\nu} = T^{\nu\mu}$. The symmetry of $T^{\mu\nu}$ is required for a proper formulation of the angular momentum conservation. The local distributions of $T^{\mu\nu}(x^\alpha)$ are important not only because they prescribe some numerical values to the energetic characteristics of the field, but also because they can be viewed responsible for the local state of motion of particles and bodies interacting with the field. In field theories governed by second-order differential equations, one expects the energy-momentum tensor to depend on squares of first-order derivatives of the field variables, but not on second derivatives.

For Lagrangian-based theories, the derivation of the conserved energy-momentum object is closely related to the variational procedure by which the equations of motion are being derived (see, for example, [2]). At the beginning it is better to speak about an energy-momentum object, rather than a tensor, because at the first steps of derivation the transformation properties are either not being discussed or not obvious. In fact, there are two routes of derivation. One produces a "canonical" object, and another produces a "metrical" object. The first route takes its origin from Euler and Lagrange. This route does not care about transformation properties of the field variables and Lagrangian itself, and whether the Lagrangian includes any metric tensor. But what is important is whether the Lagrangian contains explicitely (in a manner other than through the field variables) the independent variables (coordinates) x^α. If such dependence on x^α is present, one should not expect first integrals of the equations of motion and conserved quantities. If there is no such a dependence, some sort of conservation laws is guaranteed as a consequence of the equations of motion.

The second route is associated with the Noether identities. Here one exploits from the very beginning the transformation properties of fields and Lagrangians. One requires the action to be a quantity independent of any coordinate transformations and, hence, one requires the Lagrangian to be a scalar density, that is, a scalar function times the square root of the metric determinant. This route produces a "metrical" object, which is essentially the variational derivative of the Lagrangian with respect to the metric tensor. This object is automatically a symmetric tensor, and it is conserved if the equations of motion are satisfied. The conserved tensors are usually understood in the sense that they obey differential conservation equations, but one can also derive from them the integral conserved quantities if, as is always required, the system is isolated. For radiating systems, the fluxes of energy and momentum participate in the balance equations.

Both objects, canonical and metrical, are defined up to certain additive terms which do not violate equations of motion. These terms are a generalisation of the additive constant which arises even in a simplest one-dimensional mechanical problem, when the Lagrangian does not depend on time explicitely. It is known that the first integral of the equation of motion, which we interpret as energy, can be shifted by a constant. In field theories, the additive terms can be used to our advantage. For instance, the canonical object can be made symmetric, if it was not such originally, and the metrical object can be made free of second derivatives, if it contained them originally. Despite the different routes of derivation, the canonical and metrical objects are deeply related. If they are derived from the same Lagrangian, explicitely containing metric tensor in addition to field variables, they are equal to each other, up to a certain well defined expression calculable from the Lagrangian.

In traditional field theories, one arrives, after some work, at the energy-momentum object which is: 1) derivable from the Lagrangian in a regular prescribed way, 2) a tensor under arbitrary coordinate transformations, 3) symmetric in its components, 4) conserved due to the equations of motion obtained from the same Lagrangian, 5) free of the second (highest) derivatives of the field variables, and 6) is unique up to trivial modifications not containing the field variables. There is nothing else, in addition to these 6 conditions, that we could demand from an acceptable energy-momentum object, both on physical and mathematical grounds.

When it comes to the gravitational field, as described by the geometrical formulation of the general relativity, the things become more complicated. It is often argued that the equivalence principle forbids gravitational energy-momentum tensor. What is meant in practice is that the all first derivatives of any metric tensor $g_{\mu\nu}(x^\alpha)$ can be made, by an appropriate choice of coordinates x^α, equal to zero along the world line of a freely falling observer (along a timelike geodesic line). But the first derivatives of $g_{\mu\nu}(x^\alpha)$ can be eliminated along any world line, not necessarily of a freely falling observer. And this is true independently of the presence and form of coupling of $g_{\mu\nu}(x^\alpha)$ to other fields, and independently of whether the $g_{\mu\nu}(x^\alpha)$ obeys any equations. Since all components of a tensor can not be eliminated by a coordinate transformation, this reference to a physical principle is regarded to be an argument against a gravitational energy-momentum tensor, but the argument sounds more like a fact from the differential geometry. Despite of this argument, one usually notices that it is desirable, nevertheless, to construct at least an "effective" gravitational energy-momentum tensor. In practice, this means that we combine some terms of the Einstein equations, in one or another manner, into an object, which does not behave as a tensor even under a transition from rectangular to spherical coordinates, but which possesses some desirable properties of the energy-momentum tensor, and this is why it is an "effective" tensor. And, finally, one usually argues that the "effective" tensor becomes the "well-defined" tensor after averaging over several wavelengths. Obviously, this transmutation of a pseudotensor into a tensor can be done only in an approximate and restricted sense. And, in general, the averaging over several wavelengths means that the numerical result will depend on whether we have averaged over, say, 3 or 30 wavelengths.

This shaky situation can be tolerated as long as we are interested only in solving the Einstein equations. But this situation becomes risky when we need to know something more. It appears that the problem of a rigorously defined energy-momentum tensor may have more than a purely academic interest. We have in mind a specific question which was actually one of motivations for our renewed interest to this problem.

It is likely that the observed [3] large-angular-scale anisotropies in the microwave background radiation are caused by cosmological perturbations of quantum-mechanical origin. Cosmological perturbations can be either purely gravitational fields, as in the case of gravitational waves, or should necessarily involve gravitational component, as in the case of density perturbations. To make reliable theoretical predictions one needs to normalize the initial quantum fluctuations. In words, this means to assign energy of a half of the quantum to each mode. In practice, this implies the availability of a rigorously defined energy-momentum tensor for the field in question, which allows to enforce the energy $\frac{1}{2}\hbar\omega$, and not, say, $\frac{1}{3}\hbar\omega$ or $30\hbar\omega$, for the initial quantum state. A change in the numerical coefficient would lead to the corresponding change in the final results. The preliminary calculations show that the contributions of the quantum-mechanically produced gravitational waves and density perturbations should be approximately equal, with some preference to gravitational waves [4]. A detailed analysis of the available observational data [5] seems to favour the gravitational wave contribution twice as large as that of density perturbations. Remarkably, the factor of 2 may turn out to be important when comparing the theoretical predictions with observations. This is why, in our opinion, we cannot afford even a numerical coefficient ambiguity in such fundamental constructions as gravitational energy-momentum tensor.

We believe that the difficulty in deriving a proper gravitational energy-momentum tensor lies in the way we treat gravity, not in the nature of gravity as such. In the geometrical formulation of the general relativity, the components $g_{\mu\nu}(x^\alpha)$ play a dual role. From one side they are components of the metric tensor, from the other side they are considered gravitational field variables. If one insists on the proposition that "gravity is geometry" and "geometry is gravity", then, indeed, it is impossible to derive from the Hilbert-Einstein Lagrangian something reasonable, satisfying the 6 conditions listed above. But the geometrical approach to the general relativity is not the only one available. It is here where it is necessary to look at the general relativity from the field-theoretical positions. The general relativity can be perfectly well formulated as a strict non-linear field theory in flat space-time. This is a different formulation of the theory, not a different theory. The importance of looking at theories from different viewpoints was well emphasized by Feynman [6]: "if the peculiar viewpoint taken is truly experimentally equivalent to the usual in the realm of the known there is always a range of applications and problems in this realm for which the special viewpoint gives one a special power and clarity of thought, which is valuable in itself".

The field-theoretical formulation of the general relativity treats gravity as a non-linear tensor field $h^{\mu\nu}(x^\alpha)$ in the Minkowski space-time. In arbitrary curvilinear coordinates, the metric tensor of the flat space-time is $\gamma_{\mu\nu}(x^\alpha)$. If necessary, one is free to use the Lorentzian coordinates and to transform $\gamma_{\mu\nu}(x^\alpha)$ into the usual constant matrix $\eta_{\mu\nu}$. The Minkowski metric is not an artificially imposed "prior geometry", but a reflection of experimental facts. We know that far away from gravitating bodies, and whenever gravitational field can be neglected, the space and time intervals satisfy the requirements of the Minkowski space-time. In the presence of the gravitational field, all kinds of "rods" and "clocks" will exhibit violations of the Minkowski relationships. This is a result of the universality of the gravitational interaction (as we understand it today). One is free to interpret the results of the measurement as a manifestation of the curvature of the space-time, rather than the action of the universal gravitational field. In this sense, the Minkowski space-time becomes "unobservable". But this does not mean that the Minkowski metric is illegitimate or useless. On the contrary, it is being routinely used in relativistic astrometry and relativistic celestial mechanics. People are well aware of the general relativity and curved space-time. But it turns out to be more convenient and informative to store and analyze the data in terms of the "unobservable" flat space-time quantities (after subtraction of the theoretically calculated general-relativistic corrections), rather than in terms of directly measured "observable" quantities. If this is possible and useful in the regime of weak gravitational fields, it can be useful for any fields. In fact, for the problem of the gravitational energy-momentum tensor, the use of the Minkowski metric allows one to put everything in full order. The demonstration of this fact is the main purpose of the paper.

The structure of the paper is as follows.

In Sec. II we review definitions of the canonical and metrical energy-momentum tensors for general field theories. The associated ambiguities, and their relationship with the equations of motion, is a considerable technical complication on its own. However, we show in detail how the canonical and metrical tensors are related. The main conclusion is that, whatever the starting point, the allowed adjustments lead eventually to one and the same object satisfying the imposed requirements. We use this general analysis in Sec. IV in course of derivation of the gravitational energy-momentum tensor. Sec. III is devoted to the field-theoretical formulation of the general relativity. We start from the case of pure gravity, without matter sources. The gravitational Lagrangian and field equations are given explicitely. It is shown that the derived field equations, plus their appropriate interpretation, are fully equivalent to the Einstein equations in the geometrical formulation. In Sec. IV, being armed with the gravitational Lagrangian and field equations, we apply the general definitions of Sec. II for derivation of the gravitational energy-momentum tensor. By different routes we arrive at the energy-momentum tensor satisfying all 6 demands listed in the Abstract of the paper. It is shown that this tensor is unique up to trivial modifications which do not involve the field variables. We call this object the true energy-momentum tensor. In Sec. V we analyze the way in which the true energy-momentum tensor participates in the non-linear gravitational field equations. The gravitational energy-momentum tensor is not, and should not be, a source in the "right-hand side of Einstein's equations". But it is a source for the generalised (non-linear) d'Alembert operator. It is shown that a geometrical object most closely related to the derived energy-momentum tensor is the Landau-Lifshitz pseudotensor. Their numerical values (but not the transformation properties) are equal at least under some conditions. In Sec. VI we include matter fields in our consideration and define energy-momentum tensor for the matter fields. The gravitational energy-momentum tensor is now modified because of the presence of the matter Lagrangian. However, both, gravitational and matter energy-momentum tensors participate in the gravitational field equations at the equal footing. Their sum is the total energy-momentum tensor which is now the source for the previously mentioned generalised (non-linear) d'Alembert operator. The conservation laws for the total energy-momentum tensor are guaranteed by general theorems (Sec. II) and are manifestly satisfied as a differential consequence of the field equations. The derived equations, plus their appropriate interpretation, are fully equivalent to the Einstein's geometrical equations with matter. The final Sec. VII contains conclusions. Some technical details are relegated to Appendix A and Appendix B.

II. DEFINITIONS OF THE ENERGY-MOMENTUM TENSOR

Some of the material of this section is known in the literature but we present it in a systematic way and in a form appropriate for our further treatment of the general relativity as a field theory in flat space-time.

A. The canonical energy-momentum tensor

Let us first recall how the notion of energy arises in the simplest case of a 1-dimensional mechanical system with the Lagrangian $L = L(q, \dot{q}, t)$ and the action

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \ .$$

The equation of motion (the Euler - Lagrange equation) follows from the requirement that the action is stationary, $\delta S = 0$, under arbitrary variations of $q(t)$ vanishing at the limits of integration (what we will always assume):

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \ . \tag{1}$$

The symbol of the total derivative d/dt emphasizes the need to include the partial derivative by t if the function L depends on time explicitely. If the Lagrangian does not depend on time t explicitely, the equation (1) admits the first integral. In this case one has

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial}{\partial q}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial}{\partial \dot{q}}\left(\frac{\partial L}{\partial \dot{q}}\right)\ddot{q} \ . \tag{2}$$

By multiplying eq. (1) with $\dot{q}$ and rearranging the terms with the help of (2), one transforms eq. (1) to

$$\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = 0 \ .$$

This equation has the form of a conservation law, and the quantity $E = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$, called energy, is a constant. With the same success we could call energy the quantity $E = \left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) + C$, where C is a constant. The equation of motion (1) is still satisfied.

These considerations apply to any field theory described by the Lagrangian $L = L(q_A; q_{A,\alpha}; x^\alpha)$ where $q_A(x^\alpha)$ is a set of variables, and x^α is a set of coordinates. The variational principle produces the field equations

$$\frac{\partial L}{\partial q_A} - \left(\frac{\partial L}{\partial q_{A,\alpha}}\right)_{,\alpha} = 0, \tag{3}$$

where the last differentiation with respect to x^α includes the partial derivative by x^α, and the summation over repeated indices is (always) assumed. The field equations are conveniently written as $\frac{\delta L}{\delta q_A} = 0$, where the variational derivative δ/δ denotes (see, for example, [7])

$$\frac{\delta L(q_A; q_{A,\alpha}; x^\alpha)}{\delta q_A} \equiv \frac{\partial L}{\partial q_A} - \left(\frac{\partial L}{\partial q_{A,\alpha}}\right)_{,\alpha} . \tag{4}$$

If Lagrangian depends on second derivatives, the right-hand-side of (4) acquires an extra term, see Appendix B.

If the function L does not depend on x^α explicitely, one expects that the field equations can be transformed into the conservation equations, equal in number to the number of coordinates x^α. In this case, one has

$$L_{,\sigma} = \frac{\partial L}{\partial q_A} q_{A,\sigma} + \frac{\partial L}{\partial q_{A,\tau}} q_{A,\tau,\sigma} .$$

By multiplying eq. (3) with $q_{A,\sigma}$, taking summation over A, and making rearrangements similar to the ones described above, one obtains, as a consequence of the field equations,

$$\left(q_{A,\alpha}\frac{\partial L}{\partial q_{A,\beta}} - \delta^\beta_\alpha L\right)_{,\beta} = 0 .$$

The expression

$$\overset{c}{t}{}_\alpha^{\ \beta} = q_{A,\alpha}\frac{\partial L}{\partial q_{A,\beta}} - \delta^\beta_\alpha L$$

is the canonical (label c) conserved energy-momentum object. The upper or lower positions of α and β are not essential, but the positions of the first index (α) and the second index (β) are distinguishable. In general, the object $\overset{c}{t}{}_\alpha^{\ \beta}$ is not symmetric in α and β.

With the same success we could write for the canonical object

$$\overset{c}{t}{}_\alpha^{\ \beta} = q_{A,\alpha}\frac{\partial L}{\partial q_{A,\beta}} - \delta^\beta_\alpha L + \Psi_\alpha^{\ \beta},$$

if the function $\Psi_\alpha^{\ \beta}$ satisfies

$$\Psi_\alpha^{\ \beta}{}_{,\beta} = 0 \tag{5}$$

identically or due to the equations of motion (3). In order to satisfy (5) identically, it is sufficient to have $\Psi_\alpha^{\ \beta} = \psi_\alpha^{\ \beta\tau}{}_{,\tau}$ where $\psi_\alpha^{\ \beta\tau}$ is antisymmetric in β and τ: $\psi_\alpha^{\ \beta\tau} = -\psi_\alpha^{\ \tau\beta}$, so that $\psi_\alpha^{\ \beta\tau}{}_{,\tau,\beta} \equiv 0$. The function $\psi_\alpha^{\ \beta\tau}$ is usually called a superpotential. By an appropriate choice of $\Psi_\alpha^{\ \beta}$ one can make the object $\overset{c}{t}{}_\alpha^{\ \beta}$ symmetric in its components. The transformation properties of $\overset{c}{t}{}_\alpha^{\ \beta}$ under coordinate transformations are not defined until the transformation properties of the field variables and L are defined.

We now move to covariant relativistic theories. One normally considers physical fields of various tensor ranks (scalar, vector, tensor, etc.) in a space-time with some metric tensor. The Lagrangian L is required to be a scalar density with respect to arbitrary coordinate transformations, that is, L is a scalar function times the square root of the (minus) metric determinant. For a better contact with our further study, we consider a symmetric tensor field $h^{\mu\nu}(x^\alpha)$ placed in a flat space-time with the metric tensor $\gamma_{\mu\nu}(x^\alpha)$ written in arbitrary curvilinear coordinates x^α. The general form for the Lagrangian density is

$$L = L(\gamma^{\mu\nu}, h^{\mu\nu}, h^{\mu\nu}{}_{;\beta}) \tag{6}$$

where ";" denotes a covariant derivative defined by $\gamma_{\mu\nu}$ and the associated connection (Christoffel symbols) $C^{\alpha}_{\ \mu\nu}$. The $\gamma^{\mu\nu}$ and $C^{\alpha}_{\ \mu\nu}$ are functions of x^α but they are not dynamical variables, and hence they make the L dependent on x^α explicitely. On the general grounds, one does not expect the Euler-Lagrange equations to reduce to any conservation equations in the usual sense, i.e. in terms of vanishing partial derivatives. However, since $\gamma^{\mu\nu}{}_{;\alpha} \equiv 0$, one can derive a covariant generalisation of the conservation laws, i.e. in terms of vanishing covariant derivatives. This is, of course, consistent with our ability to choose coordinates x^α in such a way that $\gamma^{\mu\nu}$ will become a constant matrix and $C^{\alpha}_{\ \mu\nu}$ will all vanish, thus removing the explicit dependence of L on coordinates. Moreover, as we will show below, the vanishing covariant divergence will apply to the canonical energy-momentum tensor, which is now a manifestly tensorial quantity.

Let us first give a covariant generalisation to the equations of motion. The action for the Lagrangian (6) is

$$S = \frac{1}{c} \int L \, d^4x \ , \tag{7}$$

where the integral is taken over some 4-volume V. Considering $\delta h^{\mu\nu}$ and $\delta h^{\mu\nu}{}_{;\alpha}$ as independent variations we can write:

$$\delta L = \frac{\partial L}{\partial h^{\mu\nu}} \delta h^{\mu\nu} + \frac{\partial L}{\partial h^{\mu\nu}{}_{;\tau}} \delta h^{\mu\nu}{}_{;\tau} \ . \tag{8}$$

It is easy to check that the operations of variation and covariant differentiation commute. Using this property in (8) we have

$$\delta L = \frac{\partial L}{\partial h^{\mu\nu}} \delta h^{\mu\nu} + \left(\frac{\partial L}{\partial h^{\mu\nu}{}_{;\tau}} \delta h^{\mu\nu} \right)_{;\tau} - \left(\frac{\partial L}{\partial h^{\mu\nu}{}_{;\tau}} \right)_{;\tau} \delta h^{\mu\nu} \ . \tag{9}$$

Since the quantity $\frac{\partial L}{\partial h^{\mu\nu}{}_{;\tau}} \delta h^{\mu\nu}$ is a vector density of weight 1 (i.e. a vector quantity times $\sqrt{-\gamma}$) we have

$$\left(\frac{\partial L}{\partial h^{\mu\nu}{}_{;\tau}} \delta h^{\mu\nu} \right)_{;\tau} = \left(\frac{\partial L}{\partial h^{\mu\nu}{}_{;\tau}} \delta h^{\mu\nu} \right)_{,\tau} \ . \tag{10}$$

Substituting (9) into (7) and taking into account the equality above one obtains

$$\delta S = \frac{1}{c} \int \left[\frac{\partial L}{\partial h^{\mu\nu}} \delta h^{\mu\nu} - \left(\frac{\partial L}{\partial h^{\mu\nu}{}_{;\tau}} \right)_{;\tau} \delta h^{\mu\nu} + \left(\frac{\partial L}{\partial h^{\mu\nu}{}_{;\tau}} \delta h^{\mu\nu} \right)_{,\tau} \right] d^4x = 0 \ . \tag{11}$$

At the boundary of integration we have $\delta h^{\mu\nu} = 0$, so the integral of the last term in (11) is zero. The variations $\delta h^{\mu\nu}$ are arbitrary, and we arrive at the field equations in an explicitely covariant form:

$$\frac{\partial L}{\partial h^{\mu\nu}} - \left(\frac{\partial L}{\partial h^{\mu\nu}{}_{;\tau}} \right)_{;\tau} = 0 \ . \tag{12}$$

Certainly, one could have obtained the same result in a more familiar way, starting from the Lagrangian in the form containing $h^{\mu\nu}$ and the ordinary (rather than covariant) derivatives ${h^{\mu\nu}}_{,\tau}$ (see Appendix A).

One can now derive the canonical energy-momentum object in exactly the same way as was described before. Namely, one multiplies the field equations (12) by ${h^{\mu\nu}}_{;\sigma}$ and rearranges the terms to arrive at the covariant conservation law:

$$\left({h^{\mu\nu}}_{;\alpha} \frac{\partial L}{\partial {h^{\mu\nu}}_{;\beta}} - \delta^\beta_\alpha L \right)_{;\beta} = 0 \ .$$

The expression

$$\overset{c}{t}{}^{\alpha\beta} = \frac{1}{\sqrt{-\gamma}} \left(\gamma^{\alpha\tau} {h^{\mu\nu}}_{;\tau} \frac{\partial L}{\partial {h^{\mu\nu}}_{;\beta}} - \gamma^{\alpha\beta} L \right) \tag{13}$$

is the canonical energy-momentum tensor. We could also define $\overset{c}{t}{}^{\alpha\beta}$ as

$$\overset{c}{t}{}^{\alpha\beta} = \frac{1}{\sqrt{-\gamma}} \left(\gamma^{\alpha\tau} {h^{\mu\nu}}_{;\tau} \frac{\partial L}{\partial {h^{\mu\nu}}_{;\beta}} - \gamma^{\alpha\beta} L \right) + \Psi^{\alpha\beta},$$

where $\Psi^{\alpha\beta}$ is a function such that ${\Psi^{\alpha\beta}}_{;\beta} = 0$, identically or due to the field equations. The conserved canonical energy-momentum object (13) does not contain second order derivatives and is manifestly a tensorial quantity, but, in general, the canonical energy-momentum tensor is not symmetric in its components. However, it can be made symmetric by an appropriate choice of a non-symmetric $\Psi^{\alpha\beta}$.

B. The metrical energy-momentum tensor

From the general Lagrangian (6) one can also derive the metrical energy-momentum tensor. Its derivation relies on the transformation properties of all the participating quantities with respect to coordinate transformations.

An infinitesimal coordinate transformation

$$\tilde{x}^\alpha = x^\alpha - \xi^\alpha(x^\beta) \tag{14}$$

generates the Lie transformations along the vector field ξ^α, which can be presented as corresponding variations of the field variables, of the metric tensor, and of the Lagrangian: $\delta h^{\mu\nu}$, $\delta\gamma^{\mu\nu}$, and δL, respectively. Since the Lagrangian (6) is a scalar density, its variation is a total derivative

$$\delta L = (L\xi^\alpha)_{,\alpha} \ . \tag{15}$$

The change in the metric tensor is

$$\delta\gamma^{\mu\nu} = -\xi^{\mu;\nu} - \xi^{\nu;\mu} \ . \tag{16}$$

And there is also a corresponding change in the field variables

$$\delta h^{\alpha\beta} = \xi^\sigma h^{\alpha\beta}{}_{;\sigma} - h^{\alpha\sigma}\xi^\beta{}_{;\sigma} - h^{\beta\sigma}\xi^\alpha{}_{;\sigma} \tag{17}$$

but we will not need to know its concrete form for this derivation.

Taking into account (15) and assuming that the vector field ξ^α vanishes at the boundary of integration, we conclude that the variation of the action must be equal to zero:

$$\delta S = \frac{1}{c}\int \delta L \; d^4x = 0 \; . \tag{18}$$

On the other hand, we know that an arbitrary variation of L, not necessarily caused by (14), has the general form

$$\delta L = \frac{\delta L}{\delta h^{\mu\nu}}\delta h^{\mu\nu} + \frac{\delta L}{\delta \gamma^{\mu\nu}}\delta\gamma^{\mu\nu} + A^\alpha{}_{,\alpha} \tag{19}$$

where

$$A^\alpha = \frac{\partial L}{\partial h^{\mu\nu}{}_{,\alpha}}\delta h^{\mu\nu} + \frac{\partial L}{\partial \gamma^{\mu\nu}{}_{,\alpha}}\delta\gamma^{\mu\nu} \; .$$

In writing this formula we took into account the fact that the first derivatives of $\gamma^{\mu\nu}$ participate, through the Christoffel symbols, in the covariant derivatives of the field variables.

At this point we require that the field equations are satisfied

$$\frac{\delta L}{\delta h^{\mu\nu}} = 0 \tag{20}$$

so the first term in (19) is zero. In the second term of (19) we use the specific variation (16). Then, eq. (18) acquires the form

$$-2\int \left(\frac{\delta L}{\delta \gamma_{\rho\sigma}}\right)_{;\sigma} \xi_\rho \; d^4x + \int \left(2\frac{\delta L}{\delta\gamma_{\alpha\beta}}\xi_\beta + A^\alpha\right)_{,\alpha} d^4x = 0 \tag{21}$$

where we have also used the following equality:

$$\frac{\partial L}{\partial \gamma^{\alpha\beta}} = -\gamma_{\mu\alpha}\gamma_{\nu\beta}\frac{\partial L}{\partial\gamma_{\mu\nu}} \; . \tag{22}$$

The second integral in (21) transforms into a surface integral and vanishes under appropriate boundary conditions for ξ^α. Since the functions $\xi_\rho(x^\alpha)$ are arbitrary, we finally obtain:

$$-2\left(\frac{\delta L}{\delta\gamma_{\rho\sigma}}\right)_{;\sigma} = 0 \; . \tag{23}$$

The metrical (symbol m) energy-momentum tensor $\overset{m}{t}{}^{\mu\nu}$ is defined as

$$\overset{m}{t}{}^{\mu\nu} = -\frac{2}{\sqrt{-\gamma}}\frac{\delta L}{\delta\gamma_{\mu\nu}} \; , \tag{24}$$

so that eq. (23) takes the form of the covariant conservation law (valid only on solutions to the equations of motion (20)):

$$\overset{m}{t}{}^{\mu\nu}{}_{;\nu} = 0 \ . \tag{25}$$

As before, one can also write for the metrical energy-momentum tensor

$$\overset{m}{t}{}^{\mu\nu} = -\frac{2}{\sqrt{-\gamma}} \frac{\delta L}{\delta \gamma_{\mu\nu}} + \Phi^{\mu\nu} \ , \tag{26}$$

where the function $\Phi^{\mu\nu}$ satisfies $\Phi^{\mu\nu}{}_{;\nu} = 0$ identically or due to the field equations. The derived conserved object (24) is automatically symmetric and a tensor, but, as a rule, it contains second order derivatives of the field variables, even if the Lagrangian does not contain them. They are generated by one extra differentiation in the definition of the variational derivative (see the second term in eq. (4)). However, by an appropriate use of $\Phi^{\mu\nu}$ and the field equations, all second derivatives can be removed, as we will discuss in detail later on.

It is important to note that nothing in the derivation of eqs. (24), (25) actually required the $\gamma^{\mu\nu}$ to be a metric tensor of a flat space-time, that is, to have the curvature tensor constructed from $\gamma^{\mu\nu}$ equal to zero. One can still formally arrive at the equation, similar in structure to eq. (25), in arbitrary curved "background" space-time, where the covariant derivatives are now being taken with respect to the curved metric. This is an element of a field theory in the "background" space-time, which is useful in some applications (see, for example, [8], [9]). As soon as the field equations are satisfied, the corresponding covariant "conservation laws" must be valid. However, in this case, there is not and there should not be, in general, any conservation laws in the usual sense. First, one normally encounters severe integrability conditions for the field equations. The number of independent solutions, in the sense of the Cauchy problem, can be diminished, or the solutions may not exist at all. Second, the vanishing covariant divergence cannot be converted into the vanishing ordinary divergence. This is a well known formal obstacle, but it has deep and clear physical reasons: the "background" space-time is by itself a gravitational field which interacts with a system and can exchange energy with the system. For instance, even in the simplest Friedman-Robertson-Walker space-times, gravitational waves can be amplified and gravitons can be created [10].

Returning to the strictly defined energy-momentum tensors, we will now show that the canonical tensor (13) and the metrical tensor (24) are closely related.

C. Connection between metrical and canonical tensors

The metrical tensor (24) and the canonical tensor (13) are derived from the same Lagrangian (6), so one expects them to be related. To find the link between $\overset{m}{t}{}^{\mu\nu}$ and $\overset{c}{t}{}^{\mu\nu}$ we return to the derivation of $\overset{m}{t}{}^{\mu\nu}$ based on the infinitesimal transformation (14).

It is convenient to write the variation (15) in the form

$$\delta L = (L\xi^{\alpha})_{;\alpha} \ . \tag{27}$$

The replacement of the ordinary divergence by the covariant one is allowed, because the differentiated quantity $(L\xi^{\alpha})$ is a vector density. We can also write the general variation (19) in the form

$$\delta L = \frac{\delta L}{\delta h^{\alpha\beta}}\delta h^{\alpha\beta} + \frac{\delta L}{\delta \gamma^{\mu\nu}}\delta\gamma^{\mu\nu} + \left(\frac{\partial L}{\partial h^{\alpha\beta}{}_{;\tau}}\delta h^{\alpha\beta}\right)_{;\tau} + \left(\frac{\partial L}{\partial \gamma^{\mu\nu}{}_{,\tau}}\delta\gamma^{\mu\nu}\right)_{,\tau} . \tag{28}$$

In writing this expression we took into account (10) and the fact that

$$\frac{\partial L}{\partial h^{\alpha\beta}{}_{;\tau}} = \frac{\partial L}{\partial h^{\alpha\beta}{}_{,\tau}} .$$

We will now show that the differentiated quantity in the last term of (28) is also a vector density, so that the ordinary divergence can be replaced by the covariant one. Indeed, from the structure of (6) it follows that

$$\frac{\partial L}{\partial \gamma^{\mu\nu}{}_{,\tau}} = \frac{\partial L}{\partial h^{\alpha\beta}{}_{;\rho}}\frac{\partial h^{\alpha\beta}{}_{;\rho}}{\partial \gamma^{\mu\nu}{}_{,\tau}} = 2\frac{\partial L}{\partial h^{\alpha\beta}{}_{;\rho}}h^{\sigma\beta}\frac{\partial C^{\alpha}{}_{\sigma\rho}}{\partial \gamma^{\mu\nu}{}_{,\tau}} .$$

Since

$$\frac{\partial C^{\tau}{}_{\lambda\rho}}{\partial \gamma^{\alpha\beta}{}_{,\omega}} = -\frac{1}{4}(\delta^{\omega}_{\rho}\delta^{\tau}_{\alpha}\gamma_{\beta\lambda} + \delta^{\omega}_{\rho}\delta^{\tau}_{\beta}\gamma_{\alpha\lambda} + \delta^{\omega}_{\lambda}\delta^{\tau}_{\alpha}\gamma_{\beta\rho} + \delta^{\omega}_{\lambda}\delta^{\tau}_{\beta}\gamma_{\alpha\rho} - \gamma^{\tau\omega}\gamma_{\rho\alpha}\gamma_{\beta\lambda} - \gamma^{\tau\omega}\gamma_{\rho\beta}\gamma_{\alpha\lambda}) \tag{29}$$

it is now clear that the quantity $\frac{\partial L}{\partial \gamma^{\mu\nu}{}_{,\tau}}\delta\gamma^{\mu\nu}$ is a vector density. Thus, we can rewrite (28) as

$$\delta L = \frac{\delta L}{\delta h^{\alpha\beta}}\delta h^{\alpha\beta} + \frac{\delta L}{\delta \gamma^{\mu\nu}}\delta\gamma^{\mu\nu} + \left(2\frac{\partial L}{\partial h^{\alpha\beta}{}_{;\rho}}h^{\sigma\beta}\frac{\partial C^{\alpha}{}_{\sigma\rho}}{\partial \gamma^{\mu\nu}{}_{,\tau}}\delta\gamma^{\mu\nu} + \frac{\partial L}{\partial h^{\alpha\beta}{}_{;\tau}}\delta h^{\alpha\beta}\right)_{;\tau} . \tag{30}$$

The expression (30) is valid for arbitrary variations, and hence it is valid for specific variations (16), (17) caused by (14). Therefore, the difference between (27) and (30) must be equal to zero. Substituting (17) and (16) into this difference, and combining in separate groups the terms which contain ξ^{σ}, $\xi^{\sigma}{}_{;\tau}$ and $\xi^{\sigma}{}_{;\tau;\lambda}$ one obtains the equality which should be true for arbitrary vector field $\xi^{\sigma}(x^{\alpha})$:

$$\begin{gathered}
\left[\frac{\delta L}{\delta h^{\alpha\beta}}h^{\alpha\beta}{}_{;\sigma} + \left(\frac{\partial L}{\partial h^{\alpha\beta}{}_{;\tau}}h^{\alpha\beta}{}_{;\sigma} - \delta^{\tau}_{\sigma}L\right)_{;\tau}\right]\xi^{\sigma} + \\
\left[-2\frac{\delta L}{\delta \gamma^{\rho\sigma}}\gamma^{\rho\tau} + \left(\frac{\partial L}{\partial h^{\alpha\beta}{}_{;\tau}}h^{\alpha\beta}{}_{;\sigma} - \delta^{\tau}_{\sigma}L\right) - 2\frac{\delta L}{\delta h^{\alpha\sigma}}h^{\tau\alpha} + \right. \\
\left.\left(-4\frac{\partial L}{\partial h^{\alpha\beta}{}_{;\rho}}h^{\phi\beta}\frac{\partial C^{\alpha}{}_{\phi\rho}}{\partial \gamma^{\sigma\nu}{}_{,\lambda}}\gamma^{\nu\tau} - 2\frac{\partial L}{\partial h^{\alpha\sigma}{}_{;\lambda}}h^{\tau\alpha}\right)_{;\lambda}\right]\xi^{\sigma}{}_{;\tau} + \\
\left[-4\frac{\partial L}{\partial h^{\alpha\beta}{}_{;\rho}}h^{\phi\beta}\frac{\partial C^{\alpha}{}_{\phi\rho}}{\partial \gamma^{\sigma\nu}{}_{,\lambda}}\gamma^{\nu\tau} - 2\frac{\partial L}{\partial h^{\alpha\sigma}{}_{;\lambda}}h^{\tau\alpha}\right]\xi^{\sigma}{}_{;\lambda;\tau} = 0 .
\end{gathered} \tag{31}$$

The coefficient in front of ξ^{σ} is identically zero, because all the terms cancel out. To check this one has to recall the definition of the variational derivative

$$\frac{\delta L}{\delta h^{\alpha\beta}} = \frac{\partial L}{\partial h^{\alpha\beta}} - \left(\frac{\partial L}{\partial h^{\alpha\beta}{}_{,\tau}}\right)_{,\tau} ,$$

to use (A4) and (A3) for $\frac{\partial L}{\partial h^{\alpha\beta}}$ and $\left(\frac{\partial L}{\partial h^{\alpha\beta}{}_{,\tau}}\right)_{,\tau}$, and to take into account

$$\frac{\partial L}{\partial h^{\alpha\beta}} h^{\alpha\beta}{}_{;\sigma} + \frac{\partial L}{\partial h^{\alpha\beta}{}_{;\tau}} h^{\alpha\beta}{}_{;\tau;\sigma} = L_{;\sigma} \ .$$

The last term in (31), which contains $\xi^{\sigma}{}_{;\lambda;\tau}$, is also identically zero. This is true because the $\xi^{\sigma}{}_{;\lambda;\tau}$ is symmetric in the indices λ, τ whereas the coefficient is antisymmetric in these indices. To show this in detail, we denote this coefficient $\sqrt{-\gamma}\psi_{\sigma}{}^{\tau\lambda}$ and rewrite it using formula (29):

$$\sqrt{-\gamma}\psi_{\sigma}{}^{\tau\lambda} = -4\frac{\partial L}{\partial h^{\alpha\beta}{}_{;\rho}} h^{\phi\beta} \frac{\partial C^{\alpha}{}_{\phi\rho}}{\partial \gamma^{\sigma\nu}{}_{,\lambda}} \gamma^{\nu\tau} - 2\frac{\partial L}{\partial h^{\alpha\sigma}{}_{;\lambda}} h^{\tau\alpha} =$$

$$\left(\frac{\partial L}{\partial h^{\sigma\beta}{}_{;\tau}} h^{\lambda\beta} - \frac{\partial L}{\partial h^{\sigma\beta}{}_{;\lambda}} h^{\tau\beta}\right) + \frac{\partial L}{\partial h^{\alpha\beta}{}_{;\phi}} (h^{\lambda\beta}\gamma^{\alpha\tau} - h^{\tau\beta}\gamma^{\alpha\lambda})\gamma_{\phi\sigma} +$$

$$\gamma_{\phi\sigma} h^{\phi\beta} \left(\frac{\partial L}{\partial h^{\alpha\beta}{}_{;\lambda}} \gamma^{\alpha\tau} - \frac{\partial L}{\partial h^{\alpha\beta}{}_{;\tau}} \gamma^{\alpha\lambda}\right) \ . \tag{32}$$

It is now clear that $\psi_{\sigma}{}^{\tau\lambda} = -\psi_{\sigma}{}^{\lambda\tau}$. So, we are left only with the term which contains $\xi^{\sigma}{}_{;\tau}$. Since the vector field ξ^{σ} is arbitrary, this gives us the equation

$$-2\frac{\delta L}{\delta \gamma^{\rho\sigma}} \gamma^{\rho\tau} + \left(h^{\alpha\beta}{}_{;\sigma} \frac{\partial L}{\partial h^{\alpha\beta}{}_{;\tau}} - \delta^{\tau}_{\sigma} L\right) - 2\frac{\delta L}{\delta h^{\alpha\sigma}} h^{\tau\alpha} + \sqrt{-\gamma}\psi_{\sigma}{}^{\tau\lambda}{}_{;\lambda} = 0 \ .$$

Using in the first two terms the definitions of $\overset{m}{t}{}^{\mu\nu}$ and $\overset{c}{t}{}^{\mu\nu}$ and formula (22), we arrive at the universal relationship

$$-\overset{m}{t}{}^{\mu\nu} + \overset{c}{t}{}^{\mu\nu} + \psi^{\mu\nu\tau}{}_{;\tau} - \frac{2}{\sqrt{-\gamma}} \gamma^{\mu\alpha} h^{\nu\beta} \frac{\delta L}{\delta h^{\alpha\beta}} = 0. \tag{33}$$

Assuming that the field equations are satisfied (the last term vanishes) we can finally conclude that

$$\overset{m}{t}{}^{\mu\nu} = \overset{c}{t}{}^{\mu\nu} + \psi^{\mu\nu\tau}{}_{;\tau} \ .$$

Thus, the metrical and canonical tensors are related by a superpotential whose explicit form is given by eq. (32). (This derivation is similar to the one given in [11].) Obviously, the conservation laws are satisfied because $\psi^{\mu\nu\tau}{}_{;\tau;\nu} \equiv 0$.

III. FIELD THEORETICAL FORMULATION OF THE GENERAL RELATIVITY

The field theoretical approach to the general relativity treats gravity as a symmetric tensor field $h^{\mu\nu}$ in Minkowski space-time. This approach has a long and fruitful history. In fact, in the early days of special relativity, Poincare and Einstein himself started from an attempt to give a relativistic generalisation of the Newton law. Even after the acceptance of the geometrical viewpoint, various aspects of this approach have been worked out in numerous publications [12]– [28], [8], [19] – to name only a few. (One can also find references [20] useful.) We will follow a specific scheme developed in [8] and [19], as a continuation of the line of reference [28].

The gravitational field $h^{\mu\nu}(x^\alpha)$, as well as all matter fields, are defined in the Minkowski space-time with the metric tensor $\gamma_{\mu\nu}(x^\alpha)$: $d\sigma^2 = \gamma_{\mu\nu}dx^\mu dx^\nu$. The matrix $\gamma^{\mu\nu}$ is the inverse matrix to $\gamma_{\mu\nu}$, that is, $\gamma^{\alpha\beta}\gamma_{\beta\nu} = \delta^\alpha_\nu$, and γ is the determinant of the matrix $\gamma_{\mu\nu}$. The raising and lowering of indices are being performed (unless something different is explicitely stated) with the help of the metric tensor $\gamma_{\mu\nu}$. The Christoffel symbols associated with $\gamma_{\mu\nu}$ are denoted by $C^\tau{}_{\mu\nu}$, and the covariant derivatives are denoted by a semicolon ";". The curvature tensor of the Minkowski space-time is identically zero: $\breve{R}_{\alpha\beta\mu\nu}(\gamma^{\rho\sigma}) \equiv 0$.

In terms of classical mechanics, the field variables $h^{\mu\nu}$ are the generalised coordinates. Their derivatives $h^{\mu\nu}{}_{;\tau}$ (a third-rank tensor) are the generalised velocities. It is also convenient (even if not necessary) to use the generalised momenta $P^\alpha{}_{\mu\nu}$ canonically conjugated to the generalised coordinates $h^{\mu\nu}$. The object $P^\alpha_{\mu\nu}$ is a third-rank tensor, symmetric in its last indices. We will also need a contracted object $P_\alpha = P^\tau{}_{\alpha\tau} = \delta^\nu_\mu P^\mu{}_{\alpha\nu}$.

The use of $h^{\mu\nu}$ and $P^\tau{}_{\mu\nu}$ as independent variables is an element of the Hamiltonian formalism, which is also known as the first order variational formalism. We will start from this presentation, and then will consider the presentation in terms of $h^{\mu\nu}$ and $h^{\mu\nu}{}_{;\tau}$. It will be shown that the derived field equations are fully equivalent to the Einstein equations in the geometrical formulation of the general relativity.

A. Gravitational field equations in terms of generalised coordinates and momenta

The total action S of the theory consists of the gravitational part S^g and the matter part S^m: $S = S^g + S^m$. We will include the matter part in our consideration later on (Sec. VI). The action for the gravitational field is

$$S^g = \frac{1}{c}\int L^g \, d^4x \ ,$$

where the Lagrangian density L^g is

$$L^g = -\frac{\sqrt{-\gamma}}{2\kappa}\left[h^{\rho\sigma}{}_{;\alpha}P^\alpha{}_{\rho\sigma} - (\gamma^{\rho\sigma} + h^{\rho\sigma})(P^\alpha{}_{\rho\beta}P^\beta{}_{\sigma\alpha} - \frac{1}{3}P_\rho P_\sigma)\right] \tag{34}$$

and $\kappa = 8\pi G/c^4$. It is now clear that the quantities $P^\tau{}_{\mu\nu}$ are indeed the generalised momenta because

$$-\frac{\sqrt{-\gamma}}{2\kappa}P^\tau{}_{\mu\nu} = \frac{\partial L^g}{\partial h^{\mu\nu}{}_{;\tau}} = \frac{\partial L^g}{\partial h^{\mu\nu}{}_{,\tau}} \ .$$

The tensor $P^\tau_{\mu\nu}$ is related with the tensor $K^\tau{}_{\mu\nu}$ originally used in [8] by

$$P^\tau{}_{\mu\nu} = -K^\tau{}_{\mu\nu} + \frac{1}{2}\delta^\tau_\mu K_\nu + \frac{1}{2}\delta^\tau_\nu K_\mu \ .$$

To make the part of L^g, which is quadratic in the momenta $P^\tau_{\mu\nu}$, more compact, we will also write the Lagrangian in the equivalent form:

$$L^g = -\frac{\sqrt{-\gamma}}{2\kappa}\left[h^{\rho\sigma}{}_{;\alpha}P^{\alpha}_{\rho\sigma} - \frac{1}{2}\Omega^{\rho\sigma\alpha\beta}{}_{\omega\tau}P^{\tau}{}_{\rho\sigma}P^{\omega}{}_{\alpha\beta}\right] \tag{35}$$

where

$$\Omega^{\rho\sigma\alpha\beta}{}_{\omega\tau} \equiv \frac{1}{2}\Big[(\gamma^{\rho\alpha}+h^{\rho\alpha})(\delta^{\sigma}_{\omega}\delta^{\beta}_{\tau} - \frac{1}{3}\delta^{\sigma}_{\tau}\delta^{\beta}_{\omega}) + (\gamma^{\sigma\alpha}+h^{\sigma\alpha})(\delta^{\rho}_{\omega}\delta^{\beta}_{\tau} - \frac{1}{3}\delta^{\rho}_{\tau}\delta^{\beta}_{\omega}) +$$

$$(\gamma^{\rho\beta}+h^{\rho\beta})(\delta^{\sigma}_{\omega}\delta^{\alpha}_{\tau} - \frac{1}{3}\delta^{\sigma}_{\tau}\delta^{\alpha}_{\omega}) + (\gamma^{\sigma\beta}+h^{\sigma\beta})(\delta^{\rho}_{\omega}\delta^{\alpha}_{\tau} - \frac{1}{3}\delta^{\rho}_{\tau}\delta^{\alpha}_{\omega})\Big] \tag{36}$$

and $\Omega^{\mu\nu\alpha\beta}{}_{\omega\tau} = \Omega^{\nu\mu\alpha\beta}{}_{\omega\tau} = \Omega^{\mu\nu\beta\alpha}{}_{\omega\tau} = \Omega^{\alpha\beta\mu\nu}{}_{\tau\omega}$.

The gravitational field equations are derived by applying the variational principle to (34) and considering the variables $h^{\mu\nu}$ and $P^{\alpha}_{\mu\nu}$ as independent. In this framework, the field equations are

$$\frac{\partial L^g}{\partial h^{\mu\nu}} - \left(\frac{\partial L^g}{\partial h^{\mu\nu}{}_{;\tau}}\right)_{;\tau} = 0 \quad \text{and} \quad \frac{\partial L^g}{\partial P^{\alpha}_{\mu\nu}} - \left(\frac{\partial L^g}{\partial P^{\alpha}_{\mu\nu;\tau}}\right)_{;\tau} = 0 \ . \tag{37}$$

Obviously, the term $\frac{\partial L^g}{\partial P^{\alpha}_{\mu\nu;\tau}}$ in (37) is zero for the Lagrangian (34). Calculating the derivatives directly from (34) and introducing the short-hand notations for the corresponding expressions, one obtains

$$-\frac{2\kappa}{\sqrt{-\gamma}}\frac{\delta L^g}{\delta h^{\mu\nu}} \equiv r_{\mu\nu} \equiv -P^{\alpha}_{\mu\nu;\alpha} - P^{\alpha}_{\mu\beta}P^{\beta}_{\nu\alpha} + \frac{1}{3}P_{\mu}P_{\nu} = 0 \ , \tag{38}$$

$$-\frac{2\kappa}{\sqrt{-\gamma}}\frac{\delta L^g}{\delta P^{\tau}_{\mu\nu}} \equiv f_{\tau}{}^{\mu\nu} \equiv h^{\mu\nu}{}_{;\tau} - (\gamma^{\mu\alpha}+h^{\mu\alpha})P^{\nu}_{\alpha\tau} - (\gamma^{\nu\alpha}+h^{\nu\alpha})P^{\mu}_{\alpha\tau} +$$

$$\frac{1}{3}\delta^{\nu}_{\tau}(\gamma^{\mu\alpha}+h^{\mu\alpha})P_{\alpha} + \frac{1}{3}\delta^{\mu}_{\tau}(\gamma^{\nu\alpha}+h^{\nu\alpha})P_{\alpha} = 0 \ . \tag{39}$$

Using the Ω-matrix introduced above we can rewrite eq. (39) in the compact form:

$$h^{\mu\nu}{}_{;\tau} = \Omega^{\mu\nu\alpha\beta}{}_{\omega\tau}P^{\omega}_{\alpha\beta} \ . \tag{40}$$

Equations $r_{\mu\nu} = 0$ and $f_{\tau}{}^{\mu\nu} = 0$ form a complete set of equations in the framework of the first order variational formalism.

B. Field equations in terms of generalised coordinates and velocities

We will need the field equations in terms of the gravitational field variables $h^{\mu\nu}$ and their derivatives. We will derive the equations from the Lagrangian (35) written in the form containing the generalised coordinates and velocities. This is an element of the Lagrangian formalism, known also as the second order variational formalism. To implement this program one has to consider $P^{\alpha}_{\mu\nu}$ as known functions of $h^{\mu\nu}$ and $h^{\mu\nu}{}_{;\alpha}$ and to use them in the Lagrangian (35).

The link between $h^{\mu\nu}$ and $P^{\tau}{}_{\mu\nu}$ is provided by eq. (40). To solve equations (40) with respect to $P^{\tau}{}_{\mu\nu}$ we introduce the matrix $\Omega^{-1}_{\rho\sigma\mu\nu}{}^{\tau\psi}$, which is the inverse matrix to $\Omega^{\alpha\beta\mu\nu}{}_{\tau\omega}$ and satisfies the equation

$$\Omega^{\mu\nu\alpha\beta}{}_{\omega\tau}\Omega^{-1}_{\rho\sigma\mu\nu}{}^{\tau\psi} = \frac{1}{2}\delta^{\psi}_{\omega}(\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} + \delta^{\alpha}_{\sigma}\delta^{\beta}_{\rho}) \ . \tag{41}$$

The explicit form of the Ω^{-1}-matrix is not needed for the time being, but it will be given below where required. We will only use the symmetry properties of the $\Omega^{-1}_{\mu\nu\rho\sigma}{}^{\tau\omega}$ which are the same as the symmetry properties of the Ω-matrix: $\Omega^{-1}_{\mu\nu\rho\sigma}{}^{\tau\omega} = \Omega^{-1}_{\nu\mu\rho\sigma}{}^{\tau\omega} = \Omega^{-1}_{\mu\nu\sigma\rho}{}^{\tau\omega} = \Omega^{-1}_{\rho\sigma\mu\nu}{}^{\omega\tau}$. By multiplying the both sides of eq. (40) with $\Omega^{-1}_{\rho\sigma\mu\nu}{}^{\tau\psi}$ one obtains

$$P^{\tau}{}_{\mu\nu} = \Omega^{-1}_{\rho\sigma\mu\nu}{}^{\tau\omega}h^{\rho\sigma}{}_{;\omega} \ . \tag{42}$$

Now we substitute eq. (42) into eq. (35). The Lagrangian takes the elegant form

$$L^g = -\frac{\sqrt{-\gamma}}{4\kappa}\Omega^{-1}_{\rho\sigma\alpha\beta}{}^{\omega\tau}h^{\rho\sigma}{}_{;\tau}h^{\alpha\beta}{}_{;\omega} \ , \tag{43}$$

which is manifestly quadratic in the generalised velocities $h^{\mu\nu}{}_{;\tau}$. The dependence on the generalised coordinates $h^{\mu\nu}$ (as well as on the metric tensor $\gamma_{\mu\nu}$) is contained in the Ω^{-1} tensor. The Lagrangian L^g belongs to the class of Lagrangians (6) studied in Sec. II.

The field equations in the framework of the second order variational formalism are

$$\frac{\delta L^g}{\delta h^{\mu\nu}} = \frac{\partial L^g}{\partial h^{\mu\nu}} - \left(\frac{\partial L^g}{\partial h^{\mu\nu}{}_{;\tau}}\right)_{;\tau} = 0 \ .$$

In more detail, we have

$$\frac{\partial\Omega^{-1}_{\rho\sigma\alpha\beta}{}^{\omega\tau}}{\partial h^{\mu\nu}}h^{\rho\sigma}{}_{;\tau}h^{\alpha\beta}{}_{;\omega} - 2\left(\Omega^{-1}_{\mu\nu\alpha\beta}{}^{\omega\tau}h^{\alpha\beta}{}_{;\omega}\right)_{;\tau} = 0 \ .$$

The first term can be calculated by differentiating (41) with respect to $h^{\mu\nu}$ and taking into account (36). This gives

$$\frac{\partial\Omega^{-1}_{\rho\sigma\alpha\beta}{}^{\omega\tau}}{\partial h^{\mu\nu}} = -\left(\delta^{\phi}_{\pi}\delta^{\psi}_{\epsilon} - \frac{1}{3}\delta^{\phi}_{\epsilon}\delta^{\psi}_{\pi}\right)\left[\Omega^{-1}_{\mu\psi\rho\sigma}{}^{\tau\pi}\Omega^{-1}_{\nu\phi\alpha\beta}{}^{\omega\epsilon} + \Omega^{-1}_{\nu\psi\rho\sigma}{}^{\tau\pi}\Omega^{-1}_{\mu\phi\alpha\beta}{}^{\omega\epsilon}\right] \ .$$

The second term requires to recall the rules of the covariant differentiation applied to the Ω^{-1} tensor which, in turn, is a function of $h^{\mu\nu}$ and $\gamma^{\mu\nu}$:

$$\Omega^{-1}_{\mu\nu\alpha\beta}{}^{\omega\tau}{}_{;\tau} = \frac{\partial\Omega^{-1}_{\mu\nu\alpha\beta}{}^{\omega\tau}}{\partial h^{\rho\sigma}}h^{\rho\sigma}{}_{;\tau} \ .$$

Combining all together, one arrives at the field equations which are manifestly the second-order differential equations in terms of $h^{\mu\nu}$:

$$\Omega^{-1}_{\mu\nu\alpha\beta}{}^{\omega\tau}h^{\alpha\beta}{}_{;\omega;\tau} - \left(\delta^{\phi}_{\pi}\delta^{\psi}_{\epsilon} - \frac{1}{3}\delta^{\phi}_{\epsilon}\delta^{\psi}_{\pi}\right)\left[2\Omega^{-1}_{\rho\psi\mu\nu}{}^{\tau\pi}\Omega^{-1}_{\sigma\phi\alpha\beta}{}^{\omega\epsilon} - \Omega^{-1}_{\mu\psi\rho\sigma}{}^{\tau\pi}\Omega^{-1}_{\nu\phi\alpha\beta}{}^{\omega\epsilon}\right]h^{\rho\sigma}{}_{;\tau}h^{\alpha\beta}{}_{;\omega} = 0 \ . \tag{44}$$

Certainly, one arrives at exactly the same equations by substituting $P^{\tau}{}_{\mu\nu}$ found from eq. (40) (see eq.(42)) directly into (38).

C. Equivalence of the field theoretical and geometrical formulations of the general relativity

We will now show that the entire mathematical content of the general relativity (without matter sources, so far) is covered by the Lagrangian (34), or by its equivalent form (43). We will demonstrate the equivalence directly at the level of the field equations, rather than at the level of the Lagrangian (34) and its Hilbert-Einstein counterpart. The derived field equations (38), (39) can be rearranged by identical transformations into the usual Einstein equations.

First, we introduce a new tensor field $g^{\mu\nu}(x^{\alpha})$ according to the rule:

$$\sqrt{-g}g^{\mu\nu} = \sqrt{-\gamma}(\gamma^{\mu\nu} + h^{\mu\nu}) \tag{45}$$

where $g = det|g_{\mu\nu}|$ and the tensor $g_{\mu\nu}$ is the inverse matrix to the $g^{\mu\nu}$ matrix:

$$g^{\mu\alpha}g_{\nu\alpha} = \delta^{\mu}_{\nu} \ . \tag{46}$$

Let us emphasize again that the tensor $g_{\mu\nu}$ is the inverse matrix to $g^{\mu\nu}$, and not the tensor $g^{\mu\nu}$ with the lowered indices, $g_{\mu\nu} \neq \gamma_{\mu\alpha}\gamma_{\nu\beta}g^{\alpha\beta}$. For the time being, we do not assign any physical interpretation to the tensor field $g_{\mu\nu}$, we only say that the functions $g^{\mu\nu}(x^{\alpha})$ and $g_{\mu\nu}(x^{\alpha})$ are calculable from the functions $h^{\mu\nu}(x^{\alpha})$ and $\gamma^{\mu\nu}(x^{\alpha})$ according to the given rules (45), (46).

The introduced quantities allow us to write the Ω-matrix as

$$\Omega^{\rho\sigma\alpha\beta}{}_{\omega\tau} = \frac{\sqrt{-g}}{2\sqrt{-\gamma}}\left[g^{\rho\alpha}(\delta^{\sigma}_{\omega}\delta^{\beta}_{\tau} - \frac{1}{3}\delta^{\sigma}_{\tau}\delta^{\beta}_{\omega}) + g^{\sigma\alpha}(\delta^{\rho}_{\omega}\delta^{\beta}_{\tau} - \frac{1}{3}\delta^{\rho}_{\tau}\delta^{\beta}_{\omega}) + \right.$$
$$\left. g^{\rho\beta}(\delta^{\sigma}_{\omega}\delta^{\alpha}_{\tau} - \frac{1}{3}\delta^{\sigma}_{\tau}\delta^{\alpha}_{\omega}) + g^{\sigma\beta}(\delta^{\rho}_{\omega}\delta^{\alpha}_{\tau} - \frac{1}{3}\delta^{\rho}_{\tau}\delta^{\alpha}_{\omega})\right] \ .$$

We can also give the explicit form for the $\Omega^{-1}_{\mu\nu\rho\sigma}{}^{\tau\omega}$. By multiplying the both sides of (41) with $\frac{1}{4}\frac{\sqrt{-\gamma}}{\sqrt{-g}}[2\delta^{\delta}_{\alpha}(\delta^{\omega}_{\phi}\delta^{\epsilon}_{\lambda} + \delta^{\omega}_{\lambda}\delta^{\epsilon}_{\phi})g_{\epsilon\beta} - g^{\delta\omega}(2g_{\alpha\phi}g_{\beta\lambda} - g_{\alpha\beta}g_{\phi\lambda})]$ one obtains the explicit form of the Ω^{-1}-matrix:

$$\Omega^{-1}_{\mu\nu\rho\sigma}{}^{\tau\omega} = \frac{1}{4}\frac{\sqrt{-\gamma}}{\sqrt{-g}}[(\delta^{\tau}_{\mu}\delta^{\pi}_{\nu} + \delta^{\tau}_{\nu}\delta^{\pi}_{\mu})(\delta^{\omega}_{\rho}\delta^{\lambda}_{\sigma} + \delta^{\omega}_{\sigma}\delta^{\lambda}_{\rho})g_{\pi\lambda} - g^{\tau\omega}(g_{\mu\rho}g_{\nu\sigma} + g_{\nu\rho}g_{\mu\sigma} - g_{\mu\nu}g_{\rho\sigma})] \ .$$

We now want to calculate the quantity $\Gamma^{\tau}{}_{\mu\nu}$ defined by the expression

$$\Gamma^{\tau}{}_{\mu\nu} = \frac{1}{2}g^{\tau\lambda}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}) \ . \tag{47}$$

By replacing the partial derivatives with the covariant ones we get

$$\Gamma^{\tau}{}_{\mu\nu} = C^{\tau}{}_{\mu\nu} + \frac{1}{2}g^{\tau\lambda}(g_{\lambda\mu;\nu} + g_{\lambda\nu;\mu} - g_{\mu\nu;\lambda}) \ . \tag{48}$$

Now we want to trade $g_{\mu\nu;\tau}$ for $(\sqrt{-g}g^{\mu\nu})_{;\tau}$ in order to have quantities easily expressible in terms of $\gamma_{\mu\nu}$ and $h^{\alpha\beta}$. By differentiating (46) one obtains

$$g_{\mu\nu;\tau} = -g_{\mu\rho}g_{\nu\sigma}g^{\rho\sigma}{}_{;\tau} \ . \tag{49}$$

Using the formula for the differentiation of determinants, we can write

$$g^{\rho\sigma}{}_{;\tau} = \frac{1}{\sqrt{-g}}\left[(\sqrt{-g}g^{\rho\sigma})_{;\tau} - \frac{1}{2}g_{\alpha\beta}g^{\rho\sigma}(\sqrt{-g}g^{\alpha\beta})_{;\tau}\right] \ . \tag{50}$$

Substituting (49) and (50) in (48) we obtain

$$\Gamma^{\tau}{}_{\mu\nu} = C^{\tau}{}_{\mu\nu} + \frac{1}{2\sqrt{-g}}\Big[-\delta^{\tau}_{\sigma}g_{\mu\rho}(\sqrt{-g}g^{\rho\sigma})_{;\nu} - \delta^{\tau}_{\sigma}g_{\nu\rho}(\sqrt{-g}g^{\rho\sigma})_{;\mu} + g^{\tau\lambda}g_{\mu\rho}g_{\nu\sigma}(\sqrt{-g}g^{\rho\sigma})_{;\lambda} +$$
$$\frac{1}{2}g_{\alpha\beta}(\delta^{\tau}_{\mu}(\sqrt{-g}g^{\alpha\beta})_{;\nu} + \delta^{\tau}_{\nu}(\sqrt{-g}g^{\alpha\beta})_{;\mu} - g^{\tau\lambda}g_{\mu\nu}(\sqrt{-g}g^{\alpha\beta})_{;\lambda})\Big] =$$
$$\frac{1}{\sqrt{-\gamma}}\left(-\Omega^{-1}_{\mu\nu\rho\sigma}{}^{\lambda\tau} + \frac{1}{3}\delta^{\tau}_{\mu}\Omega^{-1}_{\tau\nu\rho\sigma}{}^{\lambda\tau} + \frac{1}{3}\delta^{\tau}_{\nu}\Omega^{-1}_{\tau\mu\rho\sigma}{}^{\lambda\tau}\right)(\sqrt{-g}g^{\rho\sigma})_{;\lambda} \ .$$

Finally, taking into account $(\sqrt{-g}g^{\rho\sigma})_{;\alpha} = \sqrt{-\gamma}h^{\rho\sigma}{}_{;\alpha}$ and recalling (42), we arrive at

$$\Gamma^{\tau}{}_{\mu\nu} = C^{\tau}{}_{\mu\nu} - P^{\tau}{}_{\mu\nu} + \frac{1}{3}\delta^{\tau}_{\mu}P_{\nu} + \frac{1}{3}\delta^{\tau}_{\nu}P_{\mu} \ . \tag{51}$$

Now we want to use (51) and calculate the quantity $R_{\mu\nu}$ defined by the expression

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \frac{1}{2}\Gamma^{\alpha}{}_{\mu\alpha,\nu} - \frac{1}{2}\Gamma^{\alpha}{}_{\nu\alpha,\mu} + \Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\beta}{}_{\alpha\beta} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha} \ . \tag{52}$$

The $C^{\tau}{}_{\mu\nu}$ part of $\Gamma^{\tau}{}_{\mu\nu}$ produces a series of terms which combine in the Ricci tensor $\breve{R}_{\mu\nu}$ of the flat space-time. The ordinary derivative of the tensor $P^{\tau}_{\mu\nu}$ plus all the terms containing the product of $P^{\alpha}_{\mu\beta}$ with $C^{\beta}_{\alpha\nu}$ combine in the covariant derivative of $P^{\tau}_{\mu\nu}$. All other terms produce quadratic combinations of $P^{\alpha}_{\mu\beta}$. In the result, we arrive at

$$R_{\mu\nu} = \breve{R}_{\mu\nu} - \left(P^{\alpha}{}_{\mu\nu;\alpha} + P^{\alpha}{}_{\mu\beta}P^{\beta}{}_{\nu\alpha} - \frac{1}{3}P_{\mu}P_{\nu}\right) \ . \tag{53}$$

Since $\breve{R}_{\mu\nu} \equiv 0$ we conclude that the field equations (38) are fully equivalent to the equations

$$R_{\mu\nu} = 0 \ . \tag{54}$$

The remaining step is the matter of interpretation. We can now interpret the quantities $g_{\alpha\beta}$ as the metric tensor of the curved space-time:

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \ . \tag{55}$$

Then, the quantities (47) are the Christoffel symbols associated with this metric, and the quantities (52) are the Ricci tensor of the curved space-time. Finally, equations (54) are the Einstein equations (without matter sources).

IV. THE GRAVITATIONAL ENERGY-MOMENTUM TENSOR

Being armed with the definitions of the energy-momentum tensor (Section II), as well as with the gravitational Lagrangian and field equations (Section III), we are now in the position to derive the gravitational energy-momentum tensor. We will derive both tensors, metrical and canonical, and following the general theory of their connection, we will find explicitely the superpotential which relates them. We will show that the requirement that the metrical tensor does not contain second derivatives, and the requirement that the canonical tensor is symmetric, produce one and the same object which we call the true energy-momentum tensor. This object satisfies all the 6 demands listed in the Abstract of the paper.

A. The metrical tensor

The metrical energy-momentum tensor defined by eq. (24) and derived from the Lagrangian density (34) has the following form:

$$\kappa \overset{m}{t}{}^{\mu\nu} = \frac{1}{2}\gamma^{\mu\nu} h^{\rho\sigma}{}_{;\alpha} P^{\alpha}_{\rho\sigma} + [\gamma^{\mu\rho}\gamma^{\nu\sigma} - \frac{1}{2}\gamma^{\mu\nu}(\gamma^{\rho\sigma} + h^{\rho\sigma})](P^{\alpha}_{\rho\beta} P^{\beta}_{\sigma\alpha} - \frac{1}{3} P_\rho P_\sigma) + Q^{\mu\nu}, \tag{56}$$

where

$$Q^{\mu\nu} = \frac{1}{2}(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} + \delta^{\nu}_{\rho}\delta^{\mu}_{\sigma})[-\gamma^{\rho\alpha} h^{\beta\sigma} P^{\tau}_{\alpha\beta} + (\gamma^{\alpha\tau} h^{\beta\rho} - \gamma^{\alpha\rho} h^{\beta\tau}) P^{\sigma}_{\alpha\beta}]_{;\tau} \ .$$

Expression (56) was obtained by direct calculation of the variational derivative (or alternatively, see Appendix B)

$$\frac{\delta L}{\delta \gamma_{\mu\nu}} = \frac{\partial L}{\partial \gamma_{\mu\nu}} - \left(\frac{\partial L}{\partial \gamma_{\mu\nu,\tau}}\right)_{,\tau}, \tag{57}$$

and no further rearrangements have been done. Obviously, tensor (56) is symmetric in its components, but it contains second order derivatives of $h^{\mu\nu}$ which enter the expression through the $Q^{\mu\nu}$ term. We want to single out the second derivatives of $h^{\mu\nu}$ explicitely.

By making identical transformations of the $Q^{\mu\nu}$ term one can show that the $Q^{\mu\nu}$ contains a term proportional to $r_{\mu\nu}$ and terms proportional to $f_{\tau}{}^{\mu\nu}$ and its derivatives. All these terms are equal to zero according to the field equations (38) and (39). After removing these terms, the remaining expression for $Q^{\mu\nu}$ is as follows

$$Q^{\mu\nu} = \frac{1}{2}(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} + \delta^{\nu}_{\rho}\delta^{\mu}_{\sigma}) \Big[q^{\alpha\beta\rho\sigma}(P^{\pi}_{\alpha\lambda} P^{\lambda}_{\beta\pi} - \frac{1}{3} P_\alpha P_\beta) - q^{\alpha\beta\rho\sigma}{}_{;\tau} P^{\tau}_{\alpha\beta} - \frac{1}{4}(h^{\rho\alpha}{}_{;\alpha} h^{\sigma\beta}{}_{;\beta} - h^{\rho\alpha}{}_{;\beta} h^{\sigma\beta}{}_{;\alpha}) + \frac{1}{2}(h^{\rho\tau} h^{\sigma\lambda} - h^{\tau\lambda} h^{\rho\sigma})_{;\lambda;\tau} \Big] \tag{58}$$

where

$$q^{\alpha\beta\rho\sigma} \equiv \frac{1}{2}\Big[h^{\sigma\alpha}\gamma^{\rho\beta} + h^{\rho\alpha}\gamma^{\sigma\beta} + h^{\sigma\beta}\gamma^{\rho\alpha} + h^{\rho\beta}\gamma^{\sigma\alpha} + h^{\sigma\alpha}h^{\rho\beta} + h^{\rho\alpha}h^{\sigma\beta} - h^{\rho\sigma}(\gamma^{\alpha\beta} + h^{\alpha\beta})\Big]. \tag{59}$$

The remaining expression (58), together with other terms in (56), reduce the $\kappa \overset{m}{t}{}^{\mu\nu}$ to

$$\kappa \overset{m}{t}{}^{\mu\nu}|_r = -\left[(\gamma^{\alpha\nu}+h^{\alpha\nu})(\gamma^{\beta\mu}+h^{\beta\mu}) - \frac{1}{2}(\gamma^{\alpha\beta}+h^{\alpha\beta})(\gamma^{\mu\nu}+h^{\mu\nu})\right]_{;\tau} P^{\tau}{}_{\alpha\beta} +$$
$$\left[(\gamma^{\alpha\nu}+h^{\alpha\nu})(\gamma^{\beta\mu}+h^{\beta\mu}) - \frac{1}{2}(\gamma^{\alpha\beta}+h^{\alpha\beta})(\gamma^{\mu\nu}+h^{\mu\nu})\right]\left(P^{\sigma}{}_{\beta\rho}P^{\rho}{}_{\alpha\sigma} - \frac{1}{3}P_\alpha P_\beta\right) -$$
$$\frac{1}{2}(h^{\nu\alpha}{}_{;\alpha}h^{\mu\beta}{}_{;\beta} - h^{\mu\alpha}{}_{;\beta}h^{\nu\beta}{}_{;\alpha}) + \frac{1}{4}(-2h^{\mu\nu}h^{\alpha\beta} + h^{\mu\alpha}h^{\nu\beta} + h^{\nu\alpha}h^{\mu\beta})_{;\alpha;\beta} \tag{60}$$

where the subscript $|_r$ indicates that the energy-momentum tensor was reduced on the equations of motion.

The last group of terms in (60) still contains second order derivatives of $h^{\mu\nu}$, but they all can be removed by a special choice of superpotential. Indeed, the symmetric function $\Phi^{\mu\nu}$ participating in (26) and satisfying $\Phi^{\mu\nu}{}_{;\nu} \equiv 0$ can be written as

$$\Phi^{\mu\nu} = (\phi^{\mu\nu\alpha\beta} + \phi^{\nu\mu\alpha\beta})_{;\alpha;\beta} \tag{61}$$

where

$$\phi^{\mu\nu\alpha\beta} = -\phi^{\alpha\nu\mu\beta} = -\phi^{\mu\beta\alpha\nu} = \phi^{\nu\mu\beta\alpha}. \tag{62}$$

To remove all the second order derivatives, we require

$$\frac{1}{4}(-2h^{\mu\nu}h^{\alpha\beta} + h^{\mu\alpha}h^{\nu\beta} + h^{\nu\alpha}h^{\mu\beta})_{;\alpha;\beta} + (\phi^{\mu\nu\alpha\beta} + \phi^{\nu\mu\alpha\beta})_{;\alpha;\beta} = 0. \tag{63}$$

The unique solution to this equation (up to trivial additive terms which can possibly contain $\gamma^{\mu\nu}$ but not the field variables $h^{\mu\nu}$) is:

$$\phi^{\mu\nu\alpha\beta} = \frac{1}{4}(h^{\alpha\beta}h^{\mu\nu} - h^{\alpha\nu}h^{\beta\mu}). \tag{64}$$

With the help of the superpotential (64), we can now cancel out the terms $\frac{1}{4}(-2h^{\mu\nu}h^{\alpha\beta} + h^{\mu\alpha}h^{\nu\beta} + h^{\nu\alpha}h^{\mu\beta})_{;\alpha;\beta}$. The remaining part of (60) does not contain any second order derivatives at all. To write the remaining part in a more compact form, we replace the generalised momenta by the generalised velocities with the help of (42), and use the shorter expressions $g_{\alpha\beta}$ and $g^{\alpha\beta}$ according to their definitions (45) and (46). As a result, the metrical energy-momentum tensor (56), transformed with the help of the field equations and an allowed superpotential, takes the following explicit form:

$$\kappa t^{\mu\nu} = \frac{1}{4}[2h^{\mu\nu}{}_{;\rho}h^{\rho\sigma}{}_{;\sigma} - 2h^{\mu\alpha}{}_{;\alpha}h^{\nu\beta}{}_{;\beta} + 2g^{\rho\sigma}g_{\alpha\beta}h^{\nu\beta}{}_{;\sigma}h^{\mu\alpha}{}_{;\rho} + g^{\mu\nu}g_{\alpha\rho}h^{\alpha\beta}{}_{;\sigma}h^{\rho\sigma}{}_{;\beta} -$$
$$2g^{\mu\alpha}g_{\beta\rho}h^{\nu\beta}{}_{;\sigma}h^{\rho\sigma}{}_{;\alpha} - 2g^{\nu\alpha}g_{\beta\rho}h^{\mu\beta}{}_{;\sigma}h^{\rho\sigma}{}_{;\alpha} +$$
$$\frac{1}{4}(2g^{\mu\delta}g^{\nu\omega} - g^{\mu\nu}g^{\omega\delta})(2g_{\rho\alpha}g_{\sigma\beta} - g_{\alpha\beta}g_{\rho\sigma})h^{\rho\sigma}{}_{;\delta}h^{\alpha\beta}{}_{;\omega}] \tag{65}$$

where $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are short-hand notations for the quantities (45), (46). This object is a tensor with respect to arbitrary coordinate transformations, symmetric in its components, conserved due to the field equations, free of second derivatives of $h^{\mu\nu}$, and unique up to additive terms not containing $h^{\mu\nu}$. This derivation required the use of an allowed superpotential. The last step is to show that the energy-momentum tensor (65) can also be derived according to the original definition (24), without resorting to the use of a superpotential. The tensor (65) will be derived from a modified Lagrangian, which produces exactly the same field equations as (38) and (39). This is what we will do now.

B. The constrained variational principle

Let us write the modified Lagrangian in the form

$$L^g = -\frac{\sqrt{-\gamma}}{2\kappa}\left[h^{\rho\sigma}{}_{;\alpha}P^{\alpha}_{\ \rho\sigma} - (\gamma^{\rho\sigma}+h^{\rho\sigma})(P^{\alpha}_{\ \rho\beta}P^{\beta}_{\ \sigma\alpha} - \frac{1}{3}P_\rho P_\sigma) + \Lambda^{\alpha\beta\rho\sigma}\breve{R}_{\alpha\rho\beta\sigma}\right] \tag{66}$$

where $\breve{R}_{\alpha\rho\beta\sigma}$ is the curvature tensor constructed from $\gamma_{\mu\nu}$. Obviously, we have added zero to the original Lagrangian, but this is a typical way of incorporating a constraint (in our case, $\breve{R}_{\alpha\rho\beta\sigma}=0$) by means of the undetermined Lagrange multipliers. The infinitesimal variation (16) of the metric tensor $\gamma_{\mu\nu}$ (and even its exponentiated finite version) do not change the condition $\breve{R}_{\alpha\rho\beta\sigma}=0$. The multipliers $\Lambda^{\alpha\beta\rho\sigma}$ form a tensor which depends on $\gamma^{\mu\nu}$ and $h^{\mu\nu}$ and satisfy

$$\Lambda^{\alpha\beta\rho\sigma} = -\Lambda^{\rho\beta\alpha\sigma} = -\Lambda^{\alpha\sigma\rho\beta} = \Lambda^{\beta\alpha\sigma\rho}. \tag{67}$$

The variational derivative of $\Lambda^{\alpha\beta\rho\sigma}\breve{R}_{\alpha\rho\beta\sigma}$ with respect to the metric tensor $\gamma_{\mu\nu}$ is not zero, and therefore the added term will affect the metrical energy-momentum tensor. However, the added term does not change the field equations, since the variational derivative of this term with respect to the field variables $h^{\mu\nu}$ will be multiplied by the $\breve{R}_{\alpha\rho\beta\sigma}$ and hence will vanish due to the constraint.

The metrical energy-momentum tensor (24) directly derived from (66) is now modified as compared with (56):

$$\begin{aligned}\kappa\overset{m}{t}{}^{\mu\nu}|_c = {}& \frac{1}{2}\gamma^{\mu\nu}h^{\rho\sigma}{}_{;\alpha}P^{\alpha}_{\ \rho\sigma} + [\gamma^{\mu\rho}\gamma^{\nu\sigma} - \frac{1}{2}\gamma^{\mu\nu}(\gamma^{\rho\sigma}+h^{\rho\sigma})](P^{\alpha}_{\ \rho\beta}P^{\beta}_{\ \sigma\alpha} - \frac{1}{3}P_\rho P_\sigma) + \\ & Q^{\mu\nu} - (\Lambda^{\mu\nu\alpha\beta}+\Lambda^{\nu\mu\alpha\beta})_{;\alpha;\beta}\end{aligned} \tag{68}$$

where the subscript $|_c$ indicates that the Lagrangian (66) has been used. The entire modification amounts to the last two terms (with double derivatives) in (68), which immediately suggests its connection to modifications at the expense of superpotentials (61), (62). (For a detailed derivation of the last two terms see Appendix B.) As before, the tensor $\kappa\overset{m}{t}{}^{\mu\nu}|_c$ contains second derivatives of $h^{\mu\nu}$ in the $Q^{\mu\nu}$ term. But the originally undetermined multipliers $\Lambda^{\alpha\beta\rho\sigma}$ will now be determined. They can be chosen in such a way that the remaining second derivatives of $h^{\mu\nu}$ (which could not be excluded at the field equations) can now be removed. The equations to be solved are similar to equations (63). Their unique solution is

$$\Lambda^{\mu\nu\alpha\beta} = -\frac{1}{4}(h^{\alpha\beta}h^{\mu\nu} - h^{\alpha\nu}h^{\beta\mu}).$$

Thus, the energy-momentum tensor (65) satisfies the last remaining demand: it can be derived in a regular prescribed way (24) from the Lagrangian (66).

C. The canonical tensor

The gravitational energy-momentum tensor (65) satisfying all the necessary demands has been derived along the "metrical route". We will now show that the symmetrisation procedure of the canonical tensor leads to the same object (65).

The canonical energy-momentum tensor (13) directly calculated from the Lagrangian density (43) has the form

$$\kappa \overset{c}{t}{}^{\mu\nu} = -\frac{1}{4}(2\gamma^{\nu\omega}\Omega^{-1}_{\rho\sigma\alpha\beta}{}^{\mu\tau} - \gamma^{\mu\nu}\Omega^{-1}_{\rho\sigma\alpha\beta}{}^{\omega\tau})h^{\rho\sigma}{}_{;\tau}h^{\alpha\beta}{}_{;\omega}.$$

It is convenient to use here and below the quantity $P^{\tau}{}_{\mu\nu}$ as a short-hand notation for $\Omega^{-1}_{\mu\nu\alpha\beta}{}^{\omega\tau}h^{\alpha\beta}{}_{;\omega}$ in agreement with (42). Then, the $\kappa\, \overset{c}{t}{}^{\mu\nu}$ takes the compact form

$$\kappa \overset{c}{t}{}^{\mu\nu} = -\frac{1}{2}\gamma^{\mu\tau}P^{\nu}{}_{\alpha\beta}h^{\alpha\beta}{}_{;\tau} + \frac{1}{4}\gamma^{\mu\nu}P^{\tau}{}_{\alpha\beta}h^{\alpha\beta}{}_{;\tau}. \tag{69}$$

As expected, the canonical tensor $\overset{c}{t}{}^{\mu\nu}$ is not symmetric. It can be made symmetric (see II A) by an appropriate choice of $\Psi^{\mu\nu}$. We will do this on the basis of the universal relationship (33) between the (symmetric) $\overset{m}{t}{}^{\mu\nu}$ and the (non-symmetric) $\overset{c}{t}{}^{\mu\nu}$.

The relationship in question is

$$-\kappa \overset{m}{t}{}^{\mu\nu} + \kappa \overset{c}{t}{}^{\mu\nu} + \kappa\psi^{\mu\nu\tau}{}_{;\tau} + \gamma^{\mu\alpha}h^{\nu\beta}r_{\alpha\beta} = 0 \tag{70}$$

where $\psi^{\mu\nu\tau}$ is calculated from the Lagrangian (43) according to (32):

$$\kappa\psi^{\mu\nu\tau} = \frac{1}{2}[P^{\tau}_{\alpha\beta}(\gamma^{\mu\alpha}h^{\nu\beta} - \gamma^{\alpha\nu}h^{\mu\beta}) + P^{\mu}_{\alpha\beta}(\gamma^{\alpha\tau}h^{\nu\beta} - \gamma^{\alpha\nu}h^{\tau\beta}) + P^{\nu}_{\alpha\beta}(\gamma^{\alpha\tau}h^{\mu\beta} - \gamma^{\alpha\mu}h^{\tau\beta})]. \tag{71}$$

As any superpotential does, the tensor $\psi^{\mu\nu\tau}$ satisfies the requirement $\psi^{\mu\nu\tau}{}_{;\tau;\nu} \equiv 0$. One can easily check the validity of (70) if one combines (56), (69), (71), (38) and uses the following identities:

$$(\gamma^{\alpha\beta} + h^{\alpha\beta})\left(P^{\rho}{}_{\alpha\sigma}P^{\sigma}{}_{\beta\rho} - \frac{1}{3}P_{\alpha}P_{\beta}\right) \equiv \frac{1}{2}h^{\rho\sigma}{}_{;\alpha}P^{\alpha}{}_{\rho\sigma}\ ,$$

$$(\gamma^{\nu\beta} + h^{\nu\beta})\left(P^{\rho}{}_{\alpha\sigma}P^{\sigma}{}_{\beta\rho} - \frac{1}{3}P_{\alpha}P_{\beta}\right) \equiv h^{\nu\sigma}{}_{;\rho}P^{\rho}{}_{\alpha\sigma} - \frac{1}{2}h^{\rho\sigma}{}_{;\alpha}P^{\nu}{}_{\rho\sigma}\ .$$

We now assume that the field equations are satisfied. The last term in (70) drops out. The metrical tensor $\overset{m}{t}{}^{\mu\nu}$ reduced at the equations of motion is given by (60). We need also to reduce the third term in (70) at the equations of motion. First, we differentiate the expression (71) and make identical transformations to rearrange the $\psi^{\mu\nu\tau}{}_{;\tau}$:

$$\begin{aligned}\kappa\psi^{\mu\nu\tau}{}_{;\tau} = &\left[h^{\mu\beta}(\gamma^{\nu\alpha} + h^{\nu\alpha}) - \frac{1}{2}h^{\mu\nu}(\gamma^{\alpha\beta} + h^{\alpha\beta})\right]r_{\alpha\beta} + \chi^{\mu\nu} + \\ &\frac{1}{4}(-2h^{\mu\nu}h^{\alpha\beta} + h^{\mu\alpha}h^{\nu\beta} + h^{\nu\alpha}h^{\mu\beta})_{;\alpha;\beta}\ ,\end{aligned} \tag{72}$$

where

$$\begin{aligned}\chi^{\mu\nu} = -&\left[h^{\mu\beta}(\gamma^{\nu\alpha} + h^{\nu\alpha}) - \frac{1}{2}h^{\mu\nu}(\gamma^{\alpha\beta} + h^{\alpha\beta})\right]_{;\tau}P^{\tau}{}_{\alpha\beta} + \\ &\left[h^{\mu\beta}(\gamma^{\nu\alpha} + h^{\nu\alpha}) - \frac{1}{2}h^{\mu\nu}(\gamma^{\alpha\beta} + h^{\alpha\beta})\right]_{;\tau}\left(P^{\rho}{}_{\alpha\sigma}P^{\sigma}{}_{\beta\rho} - \frac{1}{3}P_{\alpha}P_{\beta}\right) - \\ &\frac{1}{2}(h^{\nu\alpha}{}_{;\alpha}h^{\mu\beta}{}_{;\beta} - h^{\mu\alpha}{}_{;\beta}h^{\nu\beta}{}_{;\alpha})\ .\end{aligned}$$

Now we drop the term proportional to the field equations. The remaining part of $\psi^{\mu\nu\tau}{}_{;\tau}$ is

$$\kappa\psi^{\mu\nu\tau}{}_{;\tau}|_r = \chi^{\mu\nu} + \frac{1}{4}(-2h^{\mu\nu}h^{\alpha\beta} + h^{\mu\alpha}h^{\nu\beta} + h^{\nu\alpha}h^{\mu\beta})_{;\alpha;\beta} \ . \tag{73}$$

On the field equations, the original relationship (70) reduces to

$$\overset{m}{t}{}^{\mu\nu}|_r = \overset{c}{t}{}^{\mu\nu} + \ \psi^{\mu\nu\tau}{}_{;\tau}|_r. \tag{74}$$

The last term in expression (73) for $\psi^{\mu\nu\tau}{}_{;\tau}|_r$ cancels out with exactly the same term in expression (60) for $\overset{m}{t}{}^{\mu\nu}|_r$. After this cancellation, what is left on the left hand side of (74) is the metrical tensor $t^{\mu\nu}$ described by formula (65). On the right hand side of (74) we will get a symmetrised (subscript $|_s$) canonical tensor $\kappa\overset{c}{t}{}^{\mu\nu}|_s = \ \kappa\overset{c}{t}{}^{\mu\nu} + \chi^{\mu\nu}$. Note, that since $\psi^{\mu\nu\tau}{}_{;\tau;\nu} \equiv 0$, it follows from (72) that $\chi^{\mu\nu}{}_{;\nu} = 0$ on the field equations $r_{\alpha\beta} = 0$. Thus, we arrive at the equality

$$t^{\mu\nu} = \ \overset{c}{t}{}^{\mu\nu}|_s \ .$$

Since the metrical tensor (65) satisfies all the 6 demands listed above, and since it can also be obtained as a result of symmetrisation of the canonical tensor, we call it true energy-momentum tensor (and write it without any labels or subscripts).

V. GRAVITATIONAL FIELD EQUATIONS WITH GRAVITATIONAL ENERGY-MOMENTUM TENSOR

We have derived the gravitational (true) energy-momentum tensor (65) from the gravitational Lagrangian L^g according to the general definition (24). We know that the conservation laws $t^{\mu\nu}{}_{;v} = 0$ are guaranteed on solutions to the field equations. The non-linear nature of the gravitational field $h^{\mu\nu}$ makes the field a source for itself. The question arises as for how the $\kappa t^{\mu\nu}$ participates in the field equations. To answer this question one needs to rearrange the field equations and single out the $\kappa t^{\mu\nu}$ explicitely. One can proceed either from equations (38) or from equations (44). A simpler way is to take the following linear combination of the field equations (38):

$$\left[(\gamma^{\alpha\nu} + h^{\alpha\nu})(\gamma^{\beta\mu} + h^{\beta\mu}) - \frac{1}{2}(\gamma^{\alpha\beta} + h^{\alpha\beta})(\gamma^{\mu\nu} + h^{\mu\nu})\right] r_{\alpha\beta} = 0 \tag{75}$$

and to use the link (42) in order to exclude $P^{\alpha}_{\mu\nu}$. After putting all the terms in a necessary order, the field equations (75) take the following form

$$\frac{1}{2}[(\gamma^{\mu\nu} + h^{\mu\nu})(\gamma^{\alpha\beta} + h^{\alpha\beta}) - (\gamma^{\mu\alpha} + h^{\mu\alpha})(\gamma^{\nu\beta} + h^{\nu\beta})]_{;\alpha;\beta} = \frac{1}{4}[2h^{\mu\nu}{}_{;\rho}h^{\rho\sigma}{}_{;\sigma} -$$
$$2h^{\mu\alpha}{}_{;\alpha}h^{\nu\beta}{}_{;\beta} + 2g^{\rho\sigma}g_{\alpha\beta}h^{\nu\beta}{}_{;\sigma}h^{\mu\alpha}{}_{;\rho} + g^{\mu\nu}g_{\alpha\rho}h^{\alpha\beta}{}_{;\sigma}h^{\rho\sigma}{}_{;\beta} - 2g^{\mu\alpha}g_{\beta\rho}h^{\nu\beta}{}_{;\sigma}h^{\rho\sigma}{}_{;\alpha} -$$
$$2g^{\nu\alpha}g_{\beta\rho}h^{\mu\beta}{}_{;\sigma}h^{\rho\sigma}{}_{;\alpha} + \frac{1}{4}(2g^{\mu\delta}g^{\nu\omega} - g^{\mu\nu}g^{\omega\delta})(2g_{\rho\alpha}g_{\sigma\beta} - g_{\alpha\beta}g_{\rho\sigma})h^{\rho\sigma}{}_{;\delta}h^{\alpha\beta}{}_{;\omega}]. \tag{76}$$

On the right-hand side of (76) we have exactly the energy-momentum tensor (65), so the field equations can be written

$$\frac{1}{2}[(\gamma^{\mu\nu} + h^{\mu\nu})(\gamma^{\alpha\beta} + h^{\alpha\beta}) - (\gamma^{\mu\alpha} + h^{\mu\alpha})(\gamma^{\nu\beta} + h^{\nu\beta})]_{;\alpha;\beta} = \kappa t^{\mu\nu}. \tag{77}$$

The left-hand side of this equation is the generalised differential wave (d'Alembert) operator. So, the gravitational energy-momentum tensor $\kappa t^{\mu\nu}$ is not, and should not be, a source term in the "right-hand side of the Einstein equations", but it is a source term for the generalised wave operator.

Replacing the sum $(\gamma^{\mu\nu} + h^{\mu\nu})$ by the shorter expression $g^{\mu\nu}$ according to the definition (45), we can also write the gravitational field equations (77) in the form

$$\frac{1}{2}[(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})]_{;\alpha;\beta} = \kappa t^{\mu\nu} \ . \tag{78}$$

We know that the field equations (78) are fully equivalent to the Einstein equations (see Sec III). In the geometrical approach to the general relativity one interprets the quantities $g_{\mu\nu}$ as the metric tensor of a curved space-time (55). It is interesting to ask if there exists an object in the geometrical approach, which would be somehow related to the energy-momentum tensor $t^{\mu\nu}$ derived here. [The description of energy in the general relativity is, of course, a matter of a long time effort by many people who used different approaches. We would like to mention at least some of works [21]– [32], which influenced our understanding of the problem.] For this purpose we use for the first time the available coordinate freedom and introduce the Lorentzian coordinates. This means that the metric tensor $\gamma_{\mu\nu}(x^\alpha)$ is being transformed by a coordinate transformation to the usual constant matrix $\eta_{\mu\nu}$. In other words, one makes $\gamma_{00} = 1, \gamma_{11} = \gamma_{22} = \gamma_{33} = -1$, the rest of components zeros, and the determinant $\gamma = -1$. Then, all the covariant derivatives can be replaced by the ordinary ones, and all the derivatives of the metric tensor $\gamma_{\mu\nu}$ vanish. Writing the expression (65) for $t^{\mu\nu}$ in Lorentzian coordinates (subscript $|_L$) and using quantities $g^{\mu\nu}$ instead of $h^{\mu\nu}$ one finds that

$$t^{\mu\nu}|_L = (-g)t^{\mu\nu}_{LL}$$

where $t^{\mu\nu}_{LL}$ is the Landau-Lifshitz pseudotensor [2]. The field equations (78) written in Lorentzian coordinates take the form

$$\frac{1}{2}[(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})]_{,\alpha,\beta} = \kappa(-g)t^{\mu\nu}_{LL}.$$

So, the object most closely related to the derived energy-momentum tensor $t^{\mu\nu}$ is the Landau-Lifshitz pseudotensor $t^{\mu\nu}_{LL}$ times $(-g)$. Their numerical values (but not the transformation properties, unless for linear coordinate transformations) are the same at least under some conditions.

VI. GRAVITATIONAL FIELD WITH MATTER SOURCES

We will now include in our consideration matter fields interacting with the gravitational field. One or several matter fields are denoted by ϕ_A, where A is some general index.

A. Gravitational field equations and the energy-momentum tensor for the matter fields

The total action in presence of the matter sources is

$$S = \frac{1}{c} \int \left(L^g + L^m \right) d^4 x \tag{79}$$

where L^g is the gravitational Lagrangian (34) and L^m is the matter Lagrangian which includes interaction of the matter fields with the gravitational field. We assume the universal coupling of the gravitational field to all other physical fields, that is, we assume that the L^m depends on the gravitational field variables $h^{\mu\nu}$ in a specific manner:

$$L^m = L^m \left[\sqrt{-\gamma}(\gamma^{\mu\nu} + h^{\mu\nu}); (\sqrt{-\gamma}(\gamma^{\mu\nu} + h^{\mu\nu}))_{,\alpha}; \phi_A; \phi_{A,\alpha} \right]. \tag{80}$$

It was shown [8] that the L^m must depend on $h^{\mu\nu}$ and $\gamma^{\mu\nu}$ only through the combination $\sqrt{-\gamma}(\gamma^{\mu\nu} + h^{\mu\nu})$, if one wants the matter energy-momentum tensor $\tau^{\mu\nu}$ to participate in the gravitational field equations at the equal footing with the gravitational energy-momentum tensor, that is, through the total energy-momentum tensor which is the sum of the two. The matter energy-momentum tensor $\tau^{\mu\nu}$ is defined by the previously discussed (see Sec. II) universal formula

$$\tau^{\mu\nu} = -\frac{2}{\sqrt{-\gamma}} \frac{\delta L^m}{\delta \gamma_{\mu\nu}}. \tag{81}$$

Let us now turn to the derivation of the field equations. The gravitational equations are derived by applying the variational principle to the gravitational variables in the total Lagrangian. The previously derived equations (39) remain unchanged since we assume (for simplicity) that the L^m does not contain $P^\alpha_{\mu\nu}$. However, equations (38) are changed and take now the form

$$r_{\mu\nu} - \frac{2\kappa}{\sqrt{-\gamma}} \frac{\delta L^m}{\delta h^{\mu\nu}} = 0. \tag{82}$$

As for the matter field equations, they are derived by applying the variational principle to the matter variables in the total Lagrangian, which means: $\frac{\delta L^m}{\delta \phi_A} = 0$. The concrete form of the matter field equations will not be needed.

We know (Sec. V A) that equations (82) without the term caused by L^m are equivalent to equations (77), where $\kappa t^{\mu\nu}$ is given by formula (65). We want to show that the source term in the right-hand side of the gravitational equations becomes now, in presence of L^m, the total energy-momentum tensor.

Let us start from the contribution provided by L^g. Since the procedure of reduction of $\kappa \overset{m}{t}{}^{\mu\nu}$ to the final form $\kappa t^{\mu\nu}$ (65) involved the use of the equations of motion (38), which are now modified to (82), the gravitational part of the total energy-momentum tensor will also be modified, as compared with (65). Using (82) instead of (38), and getting rid of the second derivatives of $h^{\mu\nu}$ in the same way as before, one obtains

$$\kappa t^{\mu\nu}|_m = \kappa t^{\mu\nu} + q^{\alpha\beta\mu\nu} \frac{2\kappa}{\sqrt{-\gamma}} \frac{\delta L^m}{\delta h^{\alpha\beta}}$$

where the subscript $|_m$ indicates that the derivation has been done in presence of the matter fields. The $\kappa t^{\mu\nu}$ is given of course by the same formula (65), and quantities $q^{\alpha\beta\mu\nu}$ are given by formula (59). Let us now turn to $\tau^{\mu\nu}$. The universal coupling in the Lagrangian (80), that is, the fact that $h^{\mu\nu}$ and $\gamma^{\mu\nu}$ enter the L^m only in the combination $\sqrt{-\gamma}(\gamma^{\mu\nu}+h^{\mu\nu})$, allows us to relate $\frac{\delta L^m}{\delta\gamma_{\mu\nu}}$ with $\frac{\delta L^m}{\delta h^{\mu\nu}}$. After necessary transformations, one obtains

$$\tau^{\mu\nu} = \left[2\gamma^{\mu\rho}\gamma^{\nu\sigma} - \gamma^{\mu\nu}(\gamma^{\rho\sigma}+h^{\rho\sigma})\right]\frac{1}{\sqrt{-\gamma}}\frac{\delta L^m}{\delta h^{\rho\sigma}}.$$

Thus, after using the field equations and removing second derivatives of $h^{\mu\nu}$, the total energy-momentum

$$\theta^{\mu\nu} = -\frac{2}{\sqrt{-\gamma}}\frac{\delta(L^g+L^m)}{\delta\gamma_{\mu\nu}}$$

reduces to

$$\kappa(t^{\mu\nu}|_m + \tau^{\mu\nu}) = \kappa t^{\mu\nu} + \left[(\gamma^{\beta\mu}+h^{\beta\nu})(\gamma^{\alpha\nu}+h^{\alpha\nu}) - \frac{1}{2}(\gamma^{\mu\nu}+h^{\mu\nu})(\gamma^{\alpha\beta}+h^{\alpha\beta})\right]\frac{2\kappa}{\sqrt{-\gamma}}\frac{\delta L^m}{\delta h^{\alpha\beta}}.$$

Finally, we can write the gravitational field equations in the form similar to equations (77). We take the same linear combination of equations (82) as was previously done in (75). Putting all the terms in the necessary order, we arrive at the equations equivalent to (82):

$$\frac{1}{2}\left[(\gamma^{\mu\nu}+h^{\mu\nu})(\gamma^{\alpha\beta}+h^{\alpha\beta}) - (\gamma^{\mu\alpha}+h^{\mu\alpha})(\gamma^{\nu\beta}+h^{\nu\beta})\right]_{;\alpha;\beta} = \kappa(t^{\mu\nu}|_m + \tau^{\mu\nu}). \tag{83}$$

Thus, in the gravitational field equations, the total energy-momentum tensor is the source for the generalised d'Alembert operator. Obviously, the conservation laws $(t^{\mu\nu}|_m+\tau^{\mu\nu})_{;\nu} = 0$ are satisfied as a consequence of the field equations (83).

As a final remark, we should mention that the field-theoretical formulation of the general relativity allows also gauge transformations in addition to arbitrary coordinate transformations. Under gauge transformations, solutions to the field equations transform into new solutions of the same equations. In what sense and under which conditions the gauge-related solutions are physically equivalent, is a deep and nontrivial issue. This question was partially analyzed in ref. [19] but it is outside of the scope of this paper. The theory is fully consistent in its mathematical structure and physical interpretation, if the gauge transformations are applied to the gravitational field and matter variables together (even if we deal only with a couple of test particles interacting with the gravitational field and which are being used in a gedanken experiment). We mention the gauge freedom only in order to stress that all the objects and equations have been derived in arbitrary gauge, without imposing any gauge conditions.

B. Equivalence with the geometrical Einstein equations

In the geometrical approach to the general relativity one interprets the quantities $g_{\mu\nu}$, introduced by (45), (46), as the metric tensor of a curved space-time (55). The universal coupling of gravity with matter translates into

$$L^m = L^m \left[\sqrt{-g}g^{\mu\nu}; (\sqrt{-g}g^{\mu\nu})_{,\alpha}; \phi_A; \phi_{A,\alpha}\right]. \tag{84}$$

One can think of this dependence as of manifestation of the Einstein's equivalence principle.

The matter energy-momentum tensor $T_{\mu\nu}$ is now defined as the variational derivative of L^m with respect to what is now the metric tensor: $T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta L^m}{\delta g^{\mu\nu}}$ The specific form of the L^m allows us to write:

$$\frac{2}{\sqrt{-\gamma}}\frac{\delta L^m}{\delta h^{\mu\nu}} = \frac{\delta L^m}{\delta(\sqrt{-g}g^{\rho\sigma})} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}, \tag{85}$$

Tensor $T^{\mu\nu}$ certainly differs from the tensor $\tau^{\mu\nu}$ defined in the field-theoretical approach, but they are related:

$$\tau_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\gamma^{\alpha\beta}\tau_{\alpha\beta} = \left(\delta^\alpha_\mu\delta^\beta_\nu + \frac{1}{2}\gamma^{\mu\nu}h^{\alpha\beta}\right)\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}g^{\rho\sigma}T_{\rho\sigma}\right)$$

We are now in the position to prove that the field equations (83) are fully equivalent to the Einstein's geometrical equations. We know (see (53)) that

$$r_{\mu\nu} = R_{\mu\nu} - \breve{R}_{\mu\nu}. \tag{86}$$

where $\breve{R}_{\mu\nu} \equiv 0$. Combining (82) and (85) we arrive at the Einstein's geometrical equations

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}\right). \tag{87}$$

VII. CONCLUSIONS

We have shown that the field theoretical formulation of the general relativity allows us to derive the fully satisfactory gravitational energy-momentum tensor $t^{\mu\nu}$ satisfying all 6 demands listed in the Abstract of the paper. Both routes, "metrical" and "canonical", lead to one and the same unique expression (65). When the gravitational field is considered together with its matter sources, the same strict rules produce the matter energy-momentum tensor $\tau^{\mu\nu}$ and the modified gravitational energy-momentum tensor. Both tensors participate at the equal footing in the nonlinear gravitational field equations (83) which are fully equivalent to the Einstein's geometrical equations (87). These strictly defined energy-momentum tensors should be useful in practical applications.

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APPENDIX A: COVARIANT GENERALISATION OF THE EULER-LAGRANGE EQUATIONS

The covariant field equations (12) can be derived in a more traditional fashion, when one considers the field variables $h^{\mu\nu}$ and their ordinary (not covariant) derivatives ${h^{\mu\nu}}_{,\tau}$ as functions subject to variation. To emphasize this fact we rewrite the Lagrangian (6) in the form

$$L = L(\gamma^{\mu\nu}, {C^{\alpha}}_{\mu\nu}, h^{\mu\nu}, {h^{\mu\nu}}_{,\alpha}) \ . \tag{A1}$$

Starting from the Lagrangian in this form, one derives the usual field equations

$$\frac{\partial L}{\partial h^{\mu\nu}} - \left(\frac{\partial L}{\partial {h^{\mu\nu}}_{,\tau}}\right)_{,\tau} = 0 \ . \tag{A2}$$

Since the function L in (6) and (A1) is one and the same function, but written in terms of different arguments, one can relate its derivatives. The second term in (A2) transforms as follows:

$$\left(\frac{\partial L}{\partial {h^{\mu\nu}}_{,\tau}}\right)_{,\tau} = \left(\frac{\partial L}{\partial {h^{\rho\sigma}}_{;\omega}}\frac{\partial {h^{\rho\sigma}}_{;\omega}}{\partial {h^{\mu\nu}}_{,\tau}}\right)_{,\tau} = \left(\frac{\partial L}{\partial {h^{\mu\nu}}_{;\tau}}\right)_{,\tau} = \left(\frac{\partial L}{\partial {h^{\mu\nu}}_{;\tau}}\right)_{;\tau} + \frac{\partial L}{\partial {h^{\sigma\nu}}_{;\tau}}{C^{\sigma}}_{\mu\tau} + \frac{\partial L}{\partial {h^{\sigma\mu}}_{;\tau}}{C^{\sigma}}_{\nu\tau} \ . \tag{A3}$$

The first term in (A2) transforms as

$$\frac{\partial L}{\partial h^{\mu\nu}} = \frac{\partial L}{\partial h^{\mu\nu}} + \frac{\partial L}{\partial {h^{\rho\sigma}}_{;\tau}}\frac{\partial {h^{\rho\sigma}}_{;\tau}}{\partial h^{\mu\nu}} = \frac{\partial L}{\partial h^{\mu\nu}} + \frac{\partial L}{\partial {h^{\sigma\nu}}_{;\tau}}{C^{\sigma}}_{\mu\tau} + \frac{\partial L}{\partial {h^{\sigma\mu}}_{;\tau}}{C^{\sigma}}_{\nu\tau} \ . \tag{A4}$$

Using (A3) and (A4) in (A2) one obtains the required result (12). Obviously, the field equations (A2) use the Lagrangian in the form (A1), whereas the field equations (12) use the Lagrangian in the form (6).

APPENDIX B: PROOF OF EQUATION (68)

We need to show in detail that the variational derivative of the added term in Lagrangian (66), calculated at the constraint $\breve{R}_{\alpha\rho\beta\sigma} = 0$, result in the last two terms in equation (68). Let us introduce a shorter notation for the added term stressing its dependence on $\gamma_{\mu\nu}$ and derivatives:

$$\sqrt{-\gamma}\Lambda^{\alpha\beta\rho\sigma}\breve{R}_{\alpha\rho\beta\sigma} = \Pi(\gamma_{\mu\nu}; \gamma_{\mu\nu,\tau}; \gamma_{\mu\nu,\tau,\omega}).$$

Taking into account this dependence, we can write for the variation of the function Π:

$$\delta\Pi \equiv \left[\frac{\partial \Pi}{\partial \gamma_{\mu\nu}} - \left(\frac{\partial \Pi}{\partial \gamma_{\mu\nu,\tau}}\right)_{,\tau} + \left(\frac{\partial \Pi}{\partial \gamma_{\mu\nu,\tau,\omega}}\right)_{,\tau,\omega}\right]\delta\gamma_{\mu\nu} + {B^{\tau}}_{,\tau}$$

where

$$B^{\tau} = \frac{\partial \Pi}{\partial \gamma_{\mu\nu,\tau}} \delta\gamma_{\mu\nu} + \frac{\partial \Pi}{\partial \gamma_{\mu\nu,\tau,\omega}} \delta\gamma_{\mu\nu,\omega} - \left(\frac{\partial \Pi}{\partial \gamma_{\mu\nu,\tau,\omega}} \right)_{,\omega} \delta\gamma_{\mu\nu}.$$

The coefficient in front of $\delta\gamma_{\mu\nu}$ defines the variational derivative:

$$\frac{\delta \Pi}{\delta \gamma_{\mu\nu}} = \frac{\partial \Pi}{\partial \gamma_{\mu\nu}} - \left(\frac{\partial \Pi}{\partial \gamma_{\mu\nu,\tau}} \right)_{,\tau} + \left(\frac{\partial \Pi}{\partial \gamma_{\mu\nu,\tau,\omega}} \right)_{,\tau,\omega} ,$$

so that

$$\delta \Pi = \frac{\delta \Pi}{\delta \gamma_{\mu\nu}} \delta\gamma_{\mu\nu} + B^{\tau}{}_{,\tau} .$$

In order to find the required variational derivative, we can simply calculate the variation of $\sqrt{-\gamma}\Lambda^{\alpha\beta\rho\sigma}\breve{R}_{\alpha\rho\beta\sigma}$ and present it in the form

$$\delta(\sqrt{-\gamma}\Lambda^{\alpha\beta\rho\sigma}\breve{R}_{\alpha\rho\beta\sigma}) = \sqrt{-\gamma}A^{\mu\nu}\delta\gamma_{\mu\nu} + (\sqrt{-\gamma}C^{\tau})_{,\tau} .$$

The quantity $\sqrt{-\gamma}A^{\mu\nu}$ is what we need.

It is convenient to work with $\sqrt{-\gamma}\Lambda^{\alpha\beta\rho\sigma}\gamma_{\alpha\tau}\breve{R}^{\tau}{}_{\rho\beta\sigma}$ instead of $\sqrt{-\gamma}\Lambda^{\alpha\beta\rho\sigma}\breve{R}_{\alpha\rho\beta\sigma}$. The variation can be written as

$$\delta(\sqrt{-\gamma}\Lambda^{\alpha\beta\rho\sigma}\gamma_{\alpha\tau}\breve{R}^{\tau}{}_{\rho\beta\sigma}) = \delta(\sqrt{-\gamma}\Lambda^{\alpha\beta\rho\sigma}\gamma_{\alpha\tau})\breve{R}^{\tau}{}_{\rho\beta\sigma} + \delta(\breve{R}^{\tau}{}_{\rho\beta\sigma})\sqrt{-\gamma}\Lambda^{\alpha\beta\rho\sigma}\gamma_{\alpha\tau}. \tag{B1}$$

The first term on the right-hand side of (B1) vanishes due to the constraint, so we need to focus attention on the second term. The variation of the Riemann tensor is

$$\delta(\breve{R}^{\tau}{}_{\rho\beta\sigma}) = (\delta C^{\tau}{}_{\rho\sigma})_{;\beta} - (\delta C^{\tau}{}_{\rho\beta})_{;\sigma}. \tag{B2}$$

The variation of the Christoffel symbols is

$$\delta C^{\tau}{}_{\rho\sigma} = \frac{1}{2}\gamma^{\tau\lambda}(\delta\gamma_{\lambda\rho;\sigma} + \delta\gamma_{\lambda\sigma;\rho} - \delta\gamma_{\rho\sigma;\lambda}) = \frac{1}{2}(\delta^{\lambda}_{\sigma}\delta^{\alpha}_{\rho}\gamma^{\beta\tau} + \delta^{\lambda}_{\rho}\delta^{\alpha}_{\sigma}\gamma^{\beta\tau} - \delta^{\beta}_{\rho}\delta^{\alpha}_{\sigma}\gamma^{\tau\lambda})\delta\gamma_{\alpha\beta;\lambda}. \tag{B3}$$

One needs to combine (B3), (B2) and take into account properties of the symmetry of $\Lambda^{\alpha\beta\rho\sigma}$ (see (67)). After rearranging the participating terms, one gets

$$\delta(\sqrt{-\gamma}\Lambda^{\alpha\beta\rho\sigma}\breve{R}_{\alpha\rho\beta\sigma}) = -\left[\sqrt{-\gamma}(\Lambda^{\mu\nu\alpha\beta} + \Lambda^{\nu\mu\alpha\beta})_{\alpha;\beta}\right]\delta\gamma_{\mu\nu} +$$
$$\left[\sqrt{-\gamma}(\Lambda^{\alpha\phi\rho\sigma}\gamma_{\alpha\tau} - \Lambda^{\alpha\phi\rho\sigma}\gamma_{\alpha\tau})\delta C^{\tau}{}_{\rho\sigma} + \sqrt{-\gamma}(\Lambda^{\mu\nu\phi\beta} + \Lambda^{\nu\mu\phi\beta})_{;\beta}\delta\gamma_{\mu\nu}\right]_{;\phi} . \tag{B4}$$

The second term on the right-hand side of (B4) is a covariant derivative of the vector density, so the covariant derivative can be replaced by ordinary derivative, and this term has the form of $(\sqrt{-\gamma}C^{\tau})_{,\tau}$. Thus, we conclude that the sought after variational derivative, calculated at the constraint, is

$$\frac{\delta(\sqrt{-\gamma}\Lambda^{\alpha\beta\rho\sigma}\breve{R}_{\alpha\rho\beta\sigma})}{\delta\gamma_{\mu\nu}} = -\sqrt{-\gamma}(\Lambda^{\mu\nu\alpha\beta} + \Lambda^{\nu\mu\alpha\beta})_{;\alpha;\beta} .$$

Its contribution to the $\kappa \overset{m}{t}{}^{\mu\nu}|_c$ is $-(\Lambda^{\mu\nu\alpha\beta} + \Lambda^{\nu\mu\alpha\beta})_{;\alpha;\beta}$ what we needed to prove.

The calculation of (56) can be done in exactly the same way. Namely, taking the variation of the Lagrangian density (34) with respect to $\gamma_{\mu\nu}$, one obtains

$$\delta L^g = -\frac{\sqrt{-\gamma}}{2\kappa}\left\{\left[\frac{1}{2}\gamma^{\mu\nu}h^{\rho\sigma}{}_{;\alpha}P^{\alpha}_{\rho\sigma} + [\gamma^{\mu\rho}\gamma^{\nu\sigma} - \frac{1}{2}\gamma^{\mu\nu}(\gamma^{\rho\sigma} + h^{\rho\sigma})](P^{\alpha}_{\rho\beta}P^{\beta}_{\sigma\alpha} - \frac{1}{3}P_\rho P_\sigma)\right]\delta\gamma_{\mu\nu} - 2P^{\alpha}_{\rho\sigma}h^{\rho\beta}\delta C^{\sigma}{}_{\alpha\beta}\right\}. \tag{B5}$$

Using expression (B3) and rearranging the participating terms we arrive at

$$\delta L^g = -\frac{\sqrt{-\gamma}}{2\kappa}\left[\frac{1}{2}\gamma^{\mu\nu}h^{\rho\sigma}{}_{;\alpha}P^{\alpha}_{\rho\sigma} + [\gamma^{\mu\rho}\gamma^{\nu\sigma} - \frac{1}{2}\gamma^{\mu\nu}(\gamma^{\rho\sigma} + h^{\rho\sigma})](P^{\alpha}_{\rho\beta}P^{\beta}_{\sigma\alpha} - \frac{1}{3}P_\rho P_\sigma) + Q^{\mu\nu}\right]\delta\gamma_{\mu\nu} - \frac{1}{2\kappa}\left[\sqrt{-\gamma}(P^{\mu}_{\rho\sigma}h^{\rho\tau}\gamma^{\nu\sigma} + P^{\tau}_{\rho\sigma}h^{\rho\mu}\gamma^{\nu\sigma} - P^{\nu}_{\rho\sigma}h^{\rho\mu}\gamma^{\tau\sigma})\delta\gamma_{\mu\nu}\right]_{,\tau} . \tag{B6}$$

Thus, as it is stated in the text (56), the variational derivative is

$$-\frac{2\kappa}{\sqrt{-\gamma}}\frac{\delta L^g}{\delta\gamma_{\mu\nu}} = \kappa\overset{m}{t}{}^{\mu\nu} = \frac{1}{2}\gamma^{\mu\nu}h^{\rho\sigma}{}_{;\alpha}P^{\alpha}_{\rho\sigma} + [\gamma^{\mu\rho}\gamma^{\nu\sigma} - \frac{1}{2}\gamma^{\mu\nu}(\gamma^{\rho\sigma} + h^{\rho\sigma})](P^{\alpha}_{\rho\beta}P^{\beta}_{\sigma\alpha} - \frac{1}{3}P_\rho P_\sigma) + Q^{\mu\nu} . \tag{B7}$$

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